DYNAMIC PROCESSES ASSOCIATED WITH NATURAL NUMBERS

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To the memory of my parents.

Abstract. By means of a theorical development of lecture [4], we prove that dynamic processes associated to natural numbers characterize at least one arithmetic statement with temporal singularity.

Contents

1. Hyperbolic classification of Natural Numbers

For a natural number $n > 1$ the fact of being a prime is equivalent to stating that the hyperbola $xy = n$ does not contain non-trivial natural number coordinate points that is, the only natural number coordinate points in the hyperbola are $(1, n)$ and $(n, 1)$. We establish a family of bijective functions between non-negative real numbers and a half-open interval of real numbers. Bijectivity allows us to transport usual real number operations, sum

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and product, to the interval. It also allows us to deform the $xy = k$ hyperbolas with k as a real positive number in such a way that we can distinguish whether a natural number n is a prime or not by its behaviour in terms of gradients of the deformed hyperbolas near the deformed of $xy = n$ (Hyperbolic Classification of Natural Numbers).

In this section we define a function ψ which ranges from non-negative real numbers to a half-open interval, strictly increasing, continuous in $\mathbb{R}^+$ and class 1 in each interval $[m, m+1]$ $(m \in \mathbb{N} = \{0, 1, 2, 3, ...\})$. The bijectivity of ψ allows to transport the usual sum and product of $\mathbb{R}^+$ to the set $\widehat{\mathbb{R}}^+ := \psi(\mathbb{R}^+)$ in the usual manner. That is, calling $\hat{x} = \psi(x)$, we define $\hat{s} \oplus \hat{t} = \psi(s+t)$, $\hat{s} \otimes \hat{t} = \psi(st)$. Therefore, $(\widehat{\mathbb{R}}^+, \oplus, \otimes)$ is an algebraic structure isomorphic to the usual one $(\mathbb{R}^+, +, \cdot)$ and as a result, we obtain an algebraic structure $(\widehat{\mathbb{N}} := \psi(\mathbb{N}), \oplus, \otimes)$ isomorphic to the usual one $(\mathbb{N}, +, \cdot)$. The function ψ also preserves the usual orderings. Thus we transport the notation from $\mathbb{R}^+$ to $\widehat{\mathbb{R}}^+$, that is $\hat{n}$ is natural iff n is natural, $\hat{p}$ is prime iff p is prime, $\hat{x}$ is rational iff x is rational, etc. Assume that, for example $\widehat{0} = 0$, $\widehat{1} = 0'72$, $\widehat{2} = 1'3$, $\widehat{3} = 3'0001$, $\widehat{4} = \pi$, $\widehat{5} = 6'3$, $\widehat{7} = 7'21, ..., \widehat{12} = 9'03, ...$ then, the following situation would arise: the "even number" $9'03$ is the "sum" of the "prime numbers" $6'3$ and $7'21$ and the number π is the "product" of the numbers $0'72$ and π.

Obviously, until now, we have only actually changed the symbolism by means of the function ψ. If we call $\hat{x}\hat{y}$ plane the set $(\psi(\mathbb{R}^+))^2$, the hyperbolas $xy = k$ $(k > 0)$ of the xy plane with $x > 0$ and $y > 0$ are transformed by means of the function $\psi \times \psi$ at the $\hat{x} \otimes \hat{y} = \hat{k}$ "hyperbolas" of the $\hat{x}\hat{y}$ plane. We will restrict our attention to the points in the $\hat{x}\hat{y}$ plane that satisfy $\hat{x} > \hat{0}$ and $\hat{y} \geq \hat{x}$. Then, with these restrictions for the ψ function, it is possible to choose right-hand and left-hand derivatives of ψ at $m \in \mathbb{N}^* = \{1, 2, 3, ...\}$ such that we can characterize the natural number coordinate points in the $\hat{x}\hat{y}$ plane in terms of differentiability of the functions which determine the transformed hyperbolas. As a result, we can distinguish prime numbers from composite numbers in the aforementioned terms.

1.1. $\mathbb{R}^+$ coding function.

Definition 1.1. Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a map and let ψ_m be the restriction of ψ to each closed interval $[m.m+1]$ $(m \in \mathbb{N})$. We say that ψ is an $\mathbb{R}^+$ *coding function* iff: (i) $\psi(0) = 0$. (ii) $\psi \in \mathcal{C}(\mathbb{R}^+)$. (iii) $\forall m \in \mathbb{N}$, $\psi_m \in \mathcal{C}^1([m, m+1])$ with positive derivative in $[m, m+1]$.

Remarks 1.2. (1) Easily proved, if ψ is an $\mathbb{R}^+$ coding function then it is strictly increasing and consequently, injective.
(2) If $M_\psi : = \sup\{f(x) : x \in \mathbb{R}^+\}$ then, $M_\psi \in (0, +\infty]$ (being $M_\psi = +\infty$ iff ψ is not bounded), and so $\psi(\mathbb{R}^+) = [0, M_\psi)$. Therefore $\psi : \mathbb{R}^+ \to \psi(\mathbb{R}^+) = [0, M_\psi)$ is bijective, and here onwards we will refer to the ψ function as a

bijective function.
(3) We will frequently use the notation $\hat{x} = \psi(x)$. Due to the ψ bijection, we transport the sum and the product from $\mathbb{R}^+$ to $[0, M_\psi)$ in the usual manner ([3]), that is we define in $[0, M_\psi)$ the operations ψ-sum as $\hat{x} \oplus \hat{y} = \psi(x+y)$ and ψ-product as $\hat{x} \otimes \hat{y} = \psi(x \cdot y)$. Thus, $\psi : (\mathbb{R}^+, +, \cdot) \to ([0, M_\psi), \oplus, \otimes)$ is an isomorphism.
(4) The ψ function preserves the usual orderings, that is, $\hat{s} \leq \hat{t} \Leftrightarrow s \leq t,\ \hat{s} = \hat{t} \Leftrightarrow s = t$.
(5) For $\hat{x} \in [0, M_\psi)$ we say that $\hat{x}$ is a *ψ-natural number* iff x is a natural number, $\hat{x}$ is *ψ-prime* iff x is prime, $\hat{x}$ is *ψ-rational* iff x rational, etc.
(6) When we work on the set $[0, M_\psi)^2$, we say that we are on the $\hat{x}\hat{y}$ plane.
(7) For $x \geq y$ we denote $\hat{x} \sim \hat{y} := \psi(x-y) =$ (*ψ-subtraction*) and for $y \neq 0$, $\hat{x} \div \hat{y} = \psi(x/y)$ (*ψ-quotient*).

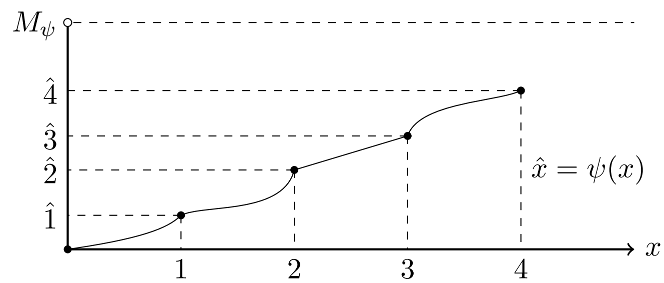


Figure 1: $\mathbb{R}^+$ coding function

1.2. **ψ-natural number coordinate points in the $\hat{x}\hat{y}$ plane.** Let ψ be an $\mathbb{R}^+$ coding function and $\alpha \in \mathbb{N}^*$. We want to characterize the (u, v) ψ-natural numbers coordinate points of the $\hat{x}\hat{y}$ plane whose coordinates ψ-sum is $\hat{\alpha}$ $(u \oplus v = \hat{\alpha})$. For this, we begin with the function $f_\alpha : [0, \alpha] \to [0, \alpha],\ f_\alpha(x) = \alpha - x$. Let us apply the function $\psi \times \psi$ to the graph

$$\Gamma(f_\alpha) = \left\{(x, y) \in (\mathbb{R}^+)^2 : x \in [0, \alpha] \wedge y = f_\alpha(x)\right\}.$$

We then obtain the transformed curve: $(\psi \times \psi)(\Gamma(f_\alpha))$. The $\hat{f}_\alpha : [0, \hat{\alpha}] \to [0, \hat{\alpha}]$ function which determines the graph of the transformed curve is

$$\hat{f}_\alpha(u) = \psi(\alpha - \psi^{-1}(u)).$$

Of course, $(u, v) \in \Gamma(\hat{f}_\alpha) = (\psi \times \psi)(\Gamma(f_\alpha))$ has ψ-natural number coordinates iff u is a ψ-natural number. The following theorem will allow a characterization of the ψ-natural coordinate points whose ψ-sum is $\hat{\alpha}$.

Theorem 1.3. *Let $\alpha \in \mathbb{N}^*$, $\psi : \mathbb{R}^+ \to [0, M_\psi)$ an $\mathbb{R}^+$ coding function and $\hat{f}_\alpha : [0, \hat{\alpha}] \to [0, \hat{\alpha}],\ \hat{f}_\alpha(u) = \psi(\alpha - \psi^{-1}(u))$. Then,*
a) $\hat{f}_\alpha$ is continuous and strictly decreasing.
b) Let $m \in \mathbb{N} : \left[\hat{m}, \hat{m} \oplus \hat{1}\right] \subset [0, \hat{\alpha}]$. Then $\hat{f}_\alpha$ is differentiable at every u

belonging to the interval $\left(\hat{m},\hat{m}\oplus\hat{1}\right)$*, with derivative*

$$\hat{f}_\alpha'(u)=-\frac{(\psi_{\alpha-m-1})'\left(\alpha-\psi^{-1}(u)\right)}{(\psi_m)'\left(\psi^{-1}(u)\right)}.$$

c) $\forall m\in\{\ 0,1,2,...,\alpha-1\ \}$*, we verify*

$$(\hat{f}_\alpha)_+'(\hat{m})=-\frac{(\psi_{\alpha-m-1})_-'(\alpha-m)}{(\psi_m)_+'(m)}.$$

d) $\forall m\in\{\ 1,2,3,...,\alpha\ \}$*, we verify*

$$(\hat{f}_\alpha)_-'(\hat{m})=-\frac{(\psi_{\alpha-m})_+'(\alpha-m)}{(\psi_{m-1})_-'(m)}.$$

Proof. a) We have $[0,\widehat{\alpha}]\xrightarrow{\psi^{-1}}[0,\alpha]\xrightarrow{f_\alpha}[0,\alpha]\xrightarrow{\psi}[0,\widehat{\alpha}]$. Therefore, $\hat{f}_\alpha=\psi\circ f_\alpha\circ\psi^{-1}$ is a composition of continuous functions, as a result it is continuous. In addition:

$$\begin{aligned}0\le s<t\le\hat{\alpha}&\Rightarrow\psi^{-1}(s)<\psi^{-1}(t)\\&\Rightarrow\alpha-\psi^{-1}(s)>\alpha-\psi^{-1}(t)\\&\Rightarrow\psi\left(\alpha-\psi^{-1}(s)\right)>\psi\left(\alpha-\psi^{-1}(t)\right)\\&\Rightarrow\hat{f}_\alpha(s)>\hat{f}_\alpha(t)\\&\Rightarrow\hat{f}_\alpha\text{ is strictly decreasing.}\end{aligned}$$

b) We have

$$\left(\hat{m},\hat{m}\oplus\hat{1}\right)\overset{\psi^{-1}}{\to}(m,m+1)\overset{f_\alpha}{\to}(\alpha-m-1,\alpha-m)\overset{\psi}{\to}\left(\hat{\alpha}\sim\hat{m}\sim\hat{1},\hat{\alpha}\sim\hat{m}\right).$$

In other words, $\hat{f}_\alpha$ maps $\hat{f}_\alpha:\left(\hat{m},\hat{m}\oplus\hat{1}\right)\to\left(\hat{\alpha}\sim\hat{m}\sim\hat{1},\hat{\alpha}\sim\hat{m}\right)$. For the $\hat{f}_\alpha$ function, all the hypotheses of the Chain Rule and Inverse Function Theorem are fulfilled, [6] thus $\forall u\in\left(\hat{m},\hat{m}\oplus\hat{1}\right)$:

$$\begin{aligned}\hat{f}_\alpha'(u)&=\psi'(\alpha-\psi^{-1}(u))\cdot\frac{-1}{\psi'(\psi^{-1}(u))}\\&=(\psi_{\alpha-m-1})'(\alpha-\psi^{-1}(u))\cdot\frac{-1}{(\psi_m)'(\psi^{-1}(u))}.\end{aligned}$$

c) Let $\epsilon>0:\hat{m}<\hat{m}\oplus\epsilon<\hat{m}\oplus\hat{1}$. As ψ is continuous and strictly increasing, there exists $0<\delta<1$ such that $\psi^{-1}(\hat{m}\oplus\varepsilon)=m+\delta$. Then,

$$[\hat{m},\hat{m}\oplus\varepsilon)\overset{\psi^{-1}}{\to}[m,m+\delta)\overset{f_\alpha}{\to}(\alpha-m-\delta,\alpha-m]\overset{\psi}{\to}\left(\hat{\alpha}\sim\hat{m}\sim\hat{\delta},\hat{\alpha}\sim\hat{m}\right].$$

Therefore $\hat{f}_\alpha$ maps $\hat{f}_\alpha:[\hat{m},\hat{m}\oplus\varepsilon)\to\left(\hat{\alpha}\sim\hat{m}\sim\hat{\delta},\hat{\alpha}\sim\hat{m}\right]$. Consequently $\forall u\in[\hat{m},\hat{m}\oplus\epsilon)$, $\hat{f}_\alpha(u)=\psi\left(\alpha-\psi^{-1}(u)\right)=\psi_{\alpha-m-1}\left(\alpha-\psi^{-1}(u)\right)$ and:

$$
\begin{aligned}
(\hat{f}_\alpha)'_+(\hat{m}) &= (\psi_{\alpha-m-1})'_-(\alpha-m)\cdot\frac{-1}{\psi'\left(\psi^{-1}(\hat{m})\right)}\\
&= -\frac{(\psi_{\alpha-m-1})'_-(\alpha-m)}{(\psi_m)'_+(m)}.
\end{aligned}
$$

d) We can similarly reason. Let $\epsilon > 0 : \hat{m} \sim \hat{1} < \hat{m} \sim \epsilon$ (or $\epsilon < \hat{1}$). Then,

$$(\hat{m} \sim \varepsilon, \hat{m}] \overset{\psi^{-1}}{\to} (m-\delta, m] \overset{f_\alpha}{\to} [\alpha - m, \alpha - m + \delta) \overset{\psi}{\to} \left[\hat{\alpha} \sim \hat{m}, \hat{\alpha} \sim \hat{m} \oplus \hat{\delta}\right)..$$

Note that $0 < \delta < 1$. Thus $\hat{f}_\alpha : (\hat{m} \sim \varepsilon, \hat{m}] \to \left[\hat{\alpha} \sim \hat{m}, \hat{\alpha} \sim \hat{m} \oplus \hat{\delta}\right)$. As a result, $\hat{f}_\alpha(u) = \psi\left(\alpha - \psi^{-1}(u)\right) = \psi_{\alpha-m}\left(\alpha - \psi^{-1}(u)\right)$. We obtain:

$$
\begin{aligned}
(\hat{f}_\alpha)'_-(\hat{m}) &= (\psi_{\alpha-m})'_+(\alpha-m)\cdot\frac{-1}{\psi'\left(\psi^{-1}(\hat{m})\right)}\\
&= -\frac{(\psi_{\alpha-m})'_+(\alpha-m)}{(\psi_{m-1})'_-(m)}.
\end{aligned}
$$

□

Let us now call $a_k = (\psi_{k-1})'_-(k)\,,\ b_k = (\psi_k)'_+(k)\ \ (k = 1, 2, 3, ...)$. From the previous theorem

$$(\hat{f}_\alpha)'_+(\hat{m}) = -\frac{a_{\alpha-m}}{b_m}\ , \quad (\hat{f}_\alpha)'_-(\hat{m}) = -\frac{b_{\alpha-m}}{a_m}.$$

Therefore, $\hat{f}_\alpha$ is differentiable for every $\hat{m}$ $\left(\hat{m} = \hat{1}, \hat{2}, ..., \hat{\alpha} \sim \hat{1}\right)$ iff

$$a_m\ a_{\alpha-m} = b_m\ b_{\alpha-m} \quad (\forall m \in \{\ 1, 2, ..., \alpha - 1\ \})\,.$$

The $\hat{f}_\alpha$ function is not differentiable for every $\hat{m}$ $\left(\hat{m} = \hat{1}, \hat{2}, \ldots, \hat{\alpha} \sim \hat{1}\right)$ iff

$$a_m\ a_{\alpha-m} \neq b_m\ b_{\alpha-m} \quad (\forall m \in \{\ 1, 2, ..., \alpha - 1\ \})\,.$$

If $\hat{f}_\alpha$ is not differentiable for every $\hat{m}$ $\left(\hat{m} = \hat{1}, \hat{2}, ..., \hat{\alpha} \sim \hat{1}\right)$, this circumstance allows us to immediately visualise the $\Gamma(\hat{f}_\alpha)$ points with ψ-natural number coordinates, and from this we may see something deeper (Fig. 2).

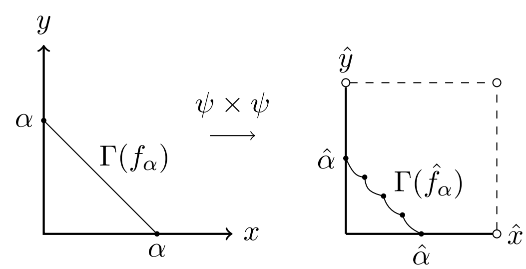


Figure 2: Identifying points with ψ-natural number coordinate points

Definition 1.4. Let $\alpha \in \mathbb{N}^*$ where $\alpha \geq 2$, and $\psi : \mathbb{R}^+ \to [0, M_\psi)$ an $\mathbb{R}^+$ coding function. It is said that the ψ function *identifies ψ-natural numbers* in $\left[\hat{0}, \hat{\alpha}\right]$ iff $\forall u \in \left(\hat{0}, \hat{\alpha}\right)$ it is verified: u is a ψ-natural number $\Leftrightarrow$ $\hat{f}_\alpha$ is not differentiable at u.

Corollary 1.5. $\alpha \in \mathbb{N}^*$, $(\alpha \geq 2)$ and $\psi : \mathbb{R}^+ \to [0, M_\psi)$ an $\mathbb{R}^+$ coding function. Then, ψ identifies ψ-natural numbers in $[0, \hat{\alpha}] \Leftrightarrow a_m \; a_{\alpha-m} \neq b_m \; b_{\alpha-m}$ $(\forall m \in \{\ 1, 2, ..., \alpha - 1\ \})$.

1.3. **ψ-hyperbolas in the $\hat{x}\hat{y}$ plane.** The aim here is to study the transformed curves of the $y = k/x$ hyperbolas $(k \in \mathbb{R}^+ - \{0\})$ by means of an $\mathbb{R}^+$ coding function in terms of differentiability. Consider the function

$$h_k : (0, +\infty) \to (0, +\infty)\,,\; h_k(x) = \frac{k}{x}.$$

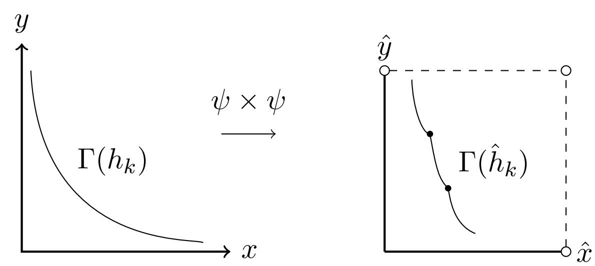


Figure 3: ψ-hyperbolas in the $\hat{x}\hat{y}$ plane

Definition 1.6. We call **ψ-hyperbola** any transformed curve graph of $\Gamma(h_k)$ by means of $\psi \times \psi$.

Notice that the function which defines the ψ-hyperbola is:

$$\hat{h}_k : (0, M_\psi) \to (0, M_\psi)\,,\; \hat{h}_k(u) = \psi\left(\frac{k}{\psi^{-1}(u)}\right).$$

Theorem 1.7. *Let $\psi : \mathbb{R}^+ \to [0, M_\psi)$ be an $\mathbb{R}^+$ coding function. Then, $\hat{h}_k : (0, M_\psi) \to (0, M_\psi)$ is continuous and strictly decreasing.*

Proof. $(0, M_\psi) \xrightarrow{\psi^{-1}} (0, +\infty) \xrightarrow{h_k} (0, +\infty) \xrightarrow{\psi} (0, M_\psi)$, thus $\hat{h}_k = \psi \circ h_k \circ \psi^{-1}$ is a composition of continuous functions, and is consequently continuous. In addition

$$\begin{aligned} 0 < s < t < M_\psi &\Rightarrow \psi^{-1}(s) < \psi^{-1}(t) \\ &\Rightarrow \frac{k}{\psi^{-1}(s)} > \frac{k}{\psi^{-1}(t)} \\ &\Rightarrow \psi\left(\frac{k}{\psi^{-1}(s)}\right) > \psi\left(\frac{k}{\psi^{-1}(t)}\right) \\ &\Rightarrow \hat{h}_k(s) > \hat{h}_k(t) \\ &\Rightarrow \hat{h}_k \text{ is strictly decreasing.} \end{aligned}$$

□

We will now analyse the differentiability of $\hat{h}_k$ distinguishing, for this, the cases in which the dependent and/or independent variable takes ψ-natural number values or not.

Theorem 1.8. *Where* $x, y \in \mathbb{R}^+ - \mathbb{N},\ \lfloor x \rfloor = n,\ \lfloor y \rfloor = m.$
1.- If $(\hat{x}, \hat{y}) \in \Gamma(\hat{h}_k)$*, then*

$$(\hat{h}_k)'(\hat{x}) = \frac{-k}{x^2} \cdot \frac{(\psi_m)'(y)}{(\psi_n)'(y)}.$$

2.- If $(\hat{x}, \hat{m}) \in \Gamma(\hat{h}_k)$*, then*

$$(\hat{h}_k)'_+ (\hat{x}) = \frac{-k}{x^2} \cdot \frac{a_m}{(\psi_n)'(x)}\ ,\ (\hat{h}_k)'_- (\hat{x}) = \frac{-k}{x^2} \cdot \frac{b_m}{(\psi_n)'(x)}.$$

3.- If $(\hat{n}, \hat{y}) \in \Gamma(\hat{h}_k)$*, then*

$$(\hat{h}_k)'_+ (\hat{n}) = \frac{-k}{n^2} \cdot \frac{(\psi_m)'(y)}{b_n}\ ,\ (\hat{h}_k)'_- (\hat{n}) = \frac{-k}{n^2} \cdot \frac{(\psi_m)'(y)}{a_n}.$$

4.- If $(\hat{n}, \hat{m}) \in \Gamma(\hat{h}_k)$*, then*

$$(\hat{h}_k)'_+ (\hat{n}) = \frac{-k}{n^2} \cdot \frac{a_m}{b_n},\ (\hat{h}_k)'_- (\hat{n}) = \frac{-k}{n^2} \cdot \frac{b_m}{a_n}.$$

Proof. Case 1 $u \in (\hat{n}, \hat{n} \oplus \hat{1})\,(n \in \mathbb{N})$ that is, u is not a ψ-natural number. We obtain $(\hat{n}, \hat{n} \oplus \hat{1}) \xrightarrow{\psi^{-1}} (n, n+1) \xrightarrow{h_k} (k/(n+1), k/n) \xrightarrow{\psi} (\hat{k} \div (\hat{n} \oplus \hat{1}), \hat{k} \div \hat{n})$ so, $\hat{h}_k$ maps $\hat{h}_k : (\hat{n}, \hat{n} \oplus \hat{1}) \to (\hat{k} \div (\hat{n} \oplus \hat{1}), \hat{k} \div \hat{n})$.
1.a) Suppose $\hat{h}_k(u)$ is not a ψ-natural number (Fig. 4). Since $k/\psi^{-1}(u)$ is not a natural number, in a neighbourhood of u , the expression of the $\hat{h}_k$ function is:

$$\hat{h}_k(t) = \psi_{\left\lfloor \frac{k}{\psi^{-1}(u)} \right\rfloor} \left(\frac{k}{\psi^{-1}(t)} \right).$$

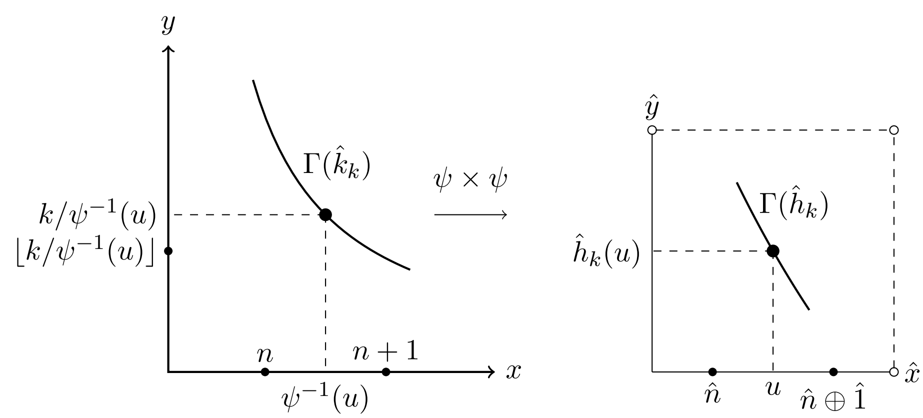


Figure 4: Finding $(\hat{h}_k)'(u)$

$$(\hat{h}_k)'(u) = \left(\psi_{\left\lfloor \frac{k}{\psi^{-1}(u)} \right\rfloor} \right)' \left(\frac{k}{\psi^{-1}(u)} \right) \cdot \frac{-k}{(\psi^{-1}(u))^2} \cdot \frac{1}{(\psi_n)'(\psi^{-1}(u))}.$$

Consequently $\hat{h}_k$ is differentiable at u.
1.b) Suppose $\hat{h}_k(u)$ is a ψ-natural number (Fig. 5). This is equivalent to

say that $k/\psi^{-1}(u)$ is a natural number. For a sufficiently small $\epsilon > 0$ we obtain

$$(u \sim \epsilon, u] \xrightarrow{\psi^{-1}} \left(\psi^{-1}(u \sim \epsilon), \psi^{-1}(u)\right] \xrightarrow{h_k}$$

$$\left[\frac{k}{\psi^{-1}(u)}, \frac{k}{\psi^{-1}(u \sim \epsilon)}\right) \xrightarrow{\psi} \left[\hat{k} \div \widehat{\psi^{-1}(u)}, \hat{k} \div \widehat{\psi^{-1}(u \sim \epsilon)}\right).$$

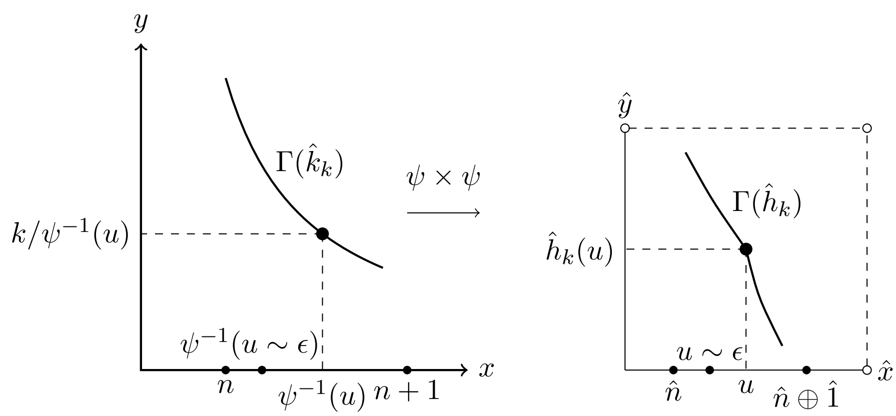


Figure 5: Finding $(\hat{h}_k)'_-(u)$

We can choose $\epsilon > 0$ such that $n < \psi^{-1}(u \sim \epsilon) < \psi^{-1}(u) < n+1$ and as a consequence for every $t \in (u \sim \epsilon, u]$ we verify $k/\psi^{-1}(u) \leq k/\psi^{-1}(t)$. That is, we can choose $\epsilon > 0$ such that $\forall t \in (u \sim \epsilon, u]$, $\hat{h}_k(t) = \psi_{\frac{k}{\psi^{-1}(u)}}\left(k/\psi^{-1}(t)\right)$. Thus:

$$(\hat{h}_k)'_-(u) = \left(\psi_{\frac{k}{\psi^{-1}(u)}}\right)'_+ \left(\frac{k}{\psi^{-1}(u)}\right) \cdot \frac{-k}{\left(\psi^{-1}(u)\right)^2} \cdot \frac{1}{(\psi_n)'\left(\psi^{-1}(u)\right)}.$$

Let us now examine the value of $(\hat{h}_k)'_+(u)$. For a sufficiently small $\epsilon > 0$ we obtain (Fig. 6)

$$[u, u \oplus \varepsilon) \xrightarrow{\psi^{-1}} \left[\psi^{-1}(u), \psi^{-1}(u \oplus \varepsilon)\right) \xrightarrow{h_k}$$

$$\left(\frac{k}{\psi^{-1}(u \oplus \epsilon)}, \frac{k}{\psi^{-1}(u)}\right] \xrightarrow{\psi} \left(\hat{k} \div \widehat{\psi^{-1}(u \oplus \epsilon)}, \hat{k} \div \widehat{\psi^{-1}(u)}\right].$$

We can choose $\epsilon > 0$ such that $n < \psi^{-1}(u) < \psi^{-1}(u \oplus \epsilon) < n+1$ and as a consequence for every $t \in [u, u \oplus \epsilon)$ we verify $k/\psi^{-1}(t) \leq k/\psi^{-1}(u)$. That is, we can choose $\epsilon > 0$ such that $\forall t \in [u, u \oplus \epsilon)$.

$$\hat{h}_k(t) = \psi_{\frac{k}{\psi^{-1}(u)} - 1}\left(\frac{k}{\psi^{-1}(t)}\right).$$

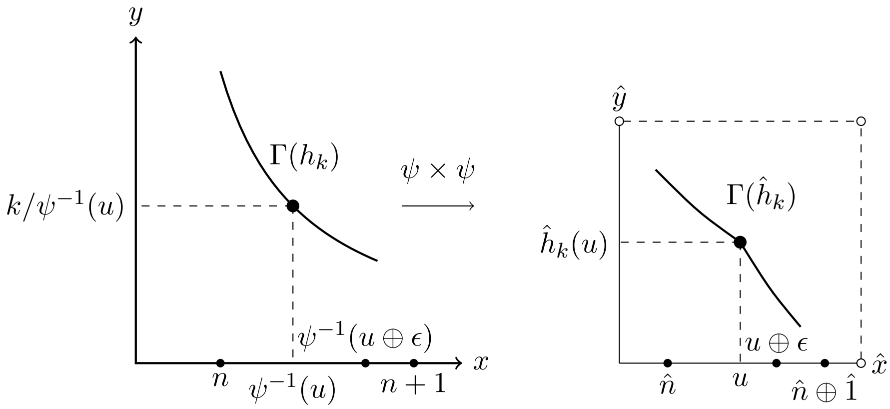


Figure 6: Finding $(\hat{h}_k)'_+(u)$

Would result:

$$(\widehat{h}_k)'_+(u) = \left(\psi_{\frac{k}{\psi^{-1}(u)}-1}\right)'_-\left(\frac{k}{\psi^{-1}(u)}\right) \cdot \frac{-k}{\left(\psi^{-1}(u)\right)^2} \cdot \frac{1}{(\psi_n)'\left(\psi^{-1}(u)\right)}.$$

Case 2 $u = \hat{n}$ $(n \in \mathbb{N}^*)$ that is, u is a ψ-natural number $(u > 0)$. For a sufficiently small $\epsilon > 0$ and $\psi(n+\delta) = \hat{n} \oplus \epsilon$ we obtain (Fig. 7)

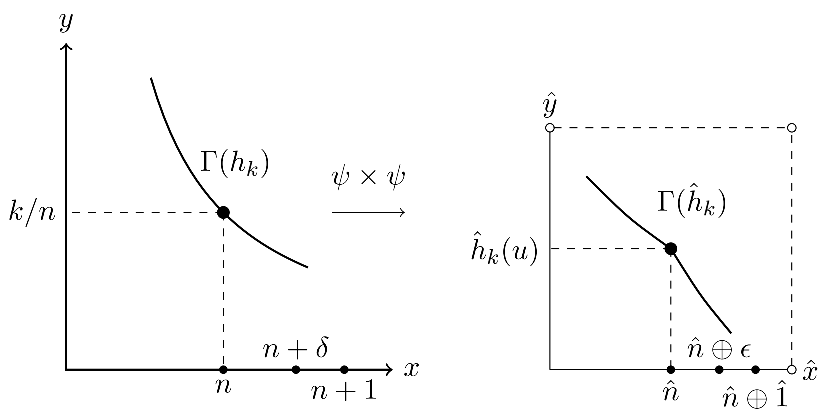


Figure 7: Finding $(\hat{h}_k)'_+(\hat{n})$

$$[\hat{n}, \hat{n} \oplus \epsilon) \overset{\psi^{-1}}{\to} [n, n+\delta) \overset{h_k}{\to} \left(\frac{k}{n+\delta}, \frac{k}{n}\right] \overset{\psi}{\to} \left(\hat{k} \div (\hat{n} \oplus \hat{\delta}), \hat{k} \div \hat{n}\right].$$

For every $t \in [\hat{n}, \hat{n} \oplus \epsilon)$, we verify $\hat{h}_k(t) = \psi_{\lfloor \frac{k}{n} \rfloor}\left(k/\psi^{-1}(t)\right)$ if $k/n \notin \mathbb{N}^*$ and $\hat{h}_k(t) = \psi_{\frac{k}{n}-1}\left(k/\psi^{-1}(t)\right)$ if $k/n \in \mathbb{N}^*$. As a consequence

$$(\hat{h}_k)'_+(\hat{n}) = \left(\psi_{\lfloor \frac{k}{n} \rfloor}\right)'\left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_n)'_+(n)} \text{ (if } k/n \notin \mathbb{N}^*),$$

$$(\hat{h}_k)'_+(\hat{n}) = \left(\psi_{\frac{k}{n}-1}\right)'_-\left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_n)'_+(n)} \text{ (if } k/n \in \mathbb{N}^*).$$

Finally we have to study the differentiability of $\hat{h}_k$ at $u = \hat{n}$ from the left side. For a sufficiently small $\epsilon > 0$ and $\psi(n-\delta) = \hat{n} \sim \epsilon$, we obtain (fig. 8)

$$(\hat{n} \sim \epsilon, \hat{n}] \overset{\psi^{-1}}{\to} (n-\delta, n] \overset{h_k}{\to} \left[\frac{k}{n}, \frac{k}{n-\delta}\right) \overset{\psi}{\to} \left[\hat{k} \div \hat{n}, \hat{k} \div (\hat{n} \sim \hat{\delta})\right).$$

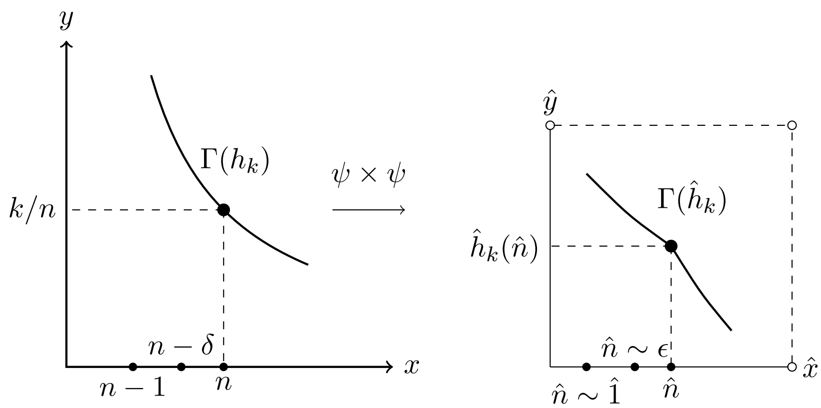


Figure 8: Finding $(\hat{h}_k)'_-(\hat{n})$

We can choose $\epsilon > 0$ such that $\forall t \in (\hat{n} \sim \epsilon, \hat{n}]$ we verify

$$\hat{h}_k\,(t) = \psi_{\lfloor \frac{k}{n} \rfloor}\left(\frac{k}{\psi^{-1}(t)}\right)$$

regardless of whether k/n is a natural number or not. This therefore would result

$$(\hat{h}_k)'_-(\hat{n}) = \left(\psi_{\lfloor \frac{k}{n} \rfloor}\right)'_+\left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_{n-1})'_-(n)}.$$

We have completed our examination of the differentiability of $\hat{h}_k$ when dependent and/or independent variables take natural ψ-natural number values or not. Since $(\psi_{i-1})'_-\,(i) = a_i$ and $(\psi_i)'_+\,(i) = b_i$ $(i = 1, 2, 3, \ldots)$, the proposition is proven. $\square$

Corollary 1.9. Let ψ be an $\mathbb{R}^+$ coding function, assume $x, y \in \mathbb{R}^+ - \mathbb{N}$, $\lfloor x \rfloor = n$, $\lfloor y \rfloor = m$ and $\hat{h}_k : (0, M_\psi) \to (0, M_\psi)$, $\hat{h}_k\,(u) = \psi\left(k/\psi^{-1}(u)\right)$. Then:
(i) If $(\hat{x}, \hat{y}) \in \Gamma(\hat{h}_k)$, then $\hat{h}_k$ is differentiable at $\hat{x}$.
(ii) If $(\hat{x}, \hat{m}) \in \Gamma(\hat{h}_k)$, then $\hat{h}_k$ is differentiable at $\hat{x}$ iff $a_m = b_m$.
(iii) If $(\hat{n}, \hat{y}) \in \Gamma(\hat{h}_k)$, then $\hat{h}_k$ is differentiable at $\hat{n}$ iff $a_n = b_n$.
(iv) If $(\hat{n}, \hat{m}) \in \Gamma(\hat{h}_k)$, then $\hat{h}_k$ is differentiable at $\hat{n}$ iff $a_n a_m = b_n b_m$.

Corollary 1.10. If we want the $\hat{h}_k$ functions to be only differentiable at the points where both the ordinate and the abscissa are not ψ-natural numbers, we must select ψ in such a way that $(a_n \neq b_n) \wedge (a_m \neq b_m) \wedge (a_n a_m \neq b_n b_m)$ or equivalently

$$a_n a_m \neq b_n b_m \,(\forall n \in \mathbb{N}^*, \forall m \in \mathbb{N}^*)\,. \tag{1.1}$$

Definition 1.11. We say that an $\mathbb{R}^+$ coding function *identifies primes* iff the $\hat{h}_k$ functions are only differentiable at the non-ψ-natural number abscissa and ordinate points

1.4. **Classification of points in the $\hat{x}\hat{y}$ plane.** Let $\psi : \mathbb{R}^+ \to [0, M_\psi)$ be an $\mathbb{R}^+$ coding function that identifies primes. The class of sets $\mathcal{H} = \{\Gamma(h_k) : k \in \mathbb{R}^+ - \{0\}\}$ is a partition of $(0, +\infty)^2$ and being ψ a bijective function, the class $\widehat{\mathcal{H}} = \{\Gamma(\hat{h}_k) : k \in \mathbb{R}^+ - \{0\}\}$ of all ψ-hyperbolas is a partition of $(0, M_\psi)^2$. Every subset of $\mathbb{R}^2$ will be considered as a topological subspace of $\mathbb{R}^2$ with the usual topology. We have the following cases:

1.- $(\hat{x}, \hat{y}) \in (0, M_\psi)^2$ $(x \notin \mathbb{N} \wedge y \notin \mathbb{N})$. Then, in a neighbourhood V of $(\hat{x}, \hat{y})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$. Of course, we mean to say the function which represents the graph of the ψ-hyperbola (Fig. 9).

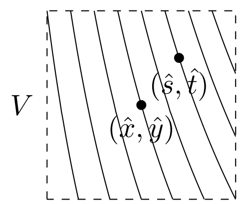


Figure 9: $x \notin \mathbb{N}, y \notin \mathbb{N}$

2.- $(\hat{x}, \hat{m}) \in (0, M_\psi)^2$ $(x \notin \mathbb{N} \wedge m \in \mathbb{N}^*)$. Then, in a neighbourhood V of $(\hat{x}, \hat{m})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{t} \neq \hat{m}$ (Fig 10).

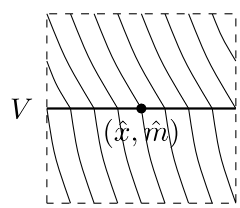


Figure 10: $x \notin \mathbb{N}, m \in \mathbb{N}^*$

3.- $(\hat{n}, \hat{y}) \in (0, M_\psi)^2$ $(n \in \mathbb{N}^* \wedge y \notin \mathbb{N})$. Then, in a neighbourhood V of $(\hat{n}, \hat{y})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{s} \neq \hat{n}$ (Fig. 11).

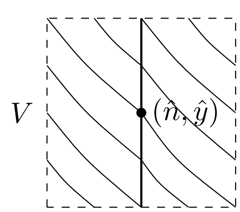


Figure 11: $n \in \mathbb{N}^*, y \notin \mathbb{N}$

4.- $(\hat{n}, \hat{m}) \in (0, M_\psi)^2$ $(n \in \mathbb{N}^* \wedge m \in \mathbb{N})$. Then, in a neighbourhood V of $(\hat{n}, \hat{m})$ we verify: $\forall(\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{s} \neq \hat{n}$ and $\hat{t} \neq \hat{m}$ (Fig. 12).

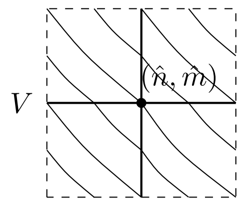


Figure 12: $n \in \mathbb{N}^*, m \in \mathbb{N}^*$

Given the symmetry of the ψ-hyperbolas with respect to the line $\hat{x} = \hat{y}$, let us consider the triangular region of the $\hat{x}\hat{y}$ plane $\mathcal{T}_\psi = \left\{(\hat{x}, \hat{y}) : \hat{y} \geq \hat{x}, \hat{x} > \hat{0}\right\}$.

Definition 1.12. Let ψ be an $\mathbb{R}^+$ coding function that identifies primes and assume that $(\hat{x}, \hat{y}) \in \mathcal{T}_\psi$. If $(\hat{x}, \hat{y}) = (\hat{n}, \hat{m})$ with $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$ we say that it is a *vortex point* with respect to ψ (Fig. 13).

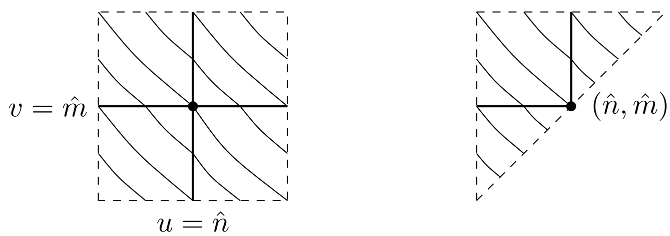


Figure 13: Vortex points

The existence of vortex points in a ψ-hyperbola allows us to identify ψ-natural numbers, only one vortex point, ψ-prime numbers. (Hyperbolic Classification of Natural Numbers)

Corollary 1.13. Let $\hat{k} \in \left(\hat{0}, M_\psi\right)$. According to the statements made above, we may classify $\hat{k}$ in terms of the behaviour of ψ-hyperbolas in $\mathcal{T}_\psi$ that are near the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$. We obtain the following classification:
1) $\hat{k}$ is a ψ-natural number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains at least a vortex point.
2) $\hat{k}$ is a ψ-prime number iff $\hat{k} \neq \hat{1}$ and the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains one and only one vortex point.
3) $\hat{k}$ is a ψ-composite number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains at least two vortex points.
4) $\hat{k}$ is not a ψ-natural number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ does not contain vortex points.

So, vortex points are characterized in terms of differentiability of the ψ-hyperbolas in $\mathcal{T}_\psi$ near these points. For every $k > 0$, denote $\bar{k} := \Gamma(\hat{h}_k) \cap \mathcal{T}_\psi$ and let $\bar{0}$ be one element different from $\bar{k}$ $(k > 0)$. Define $\mathfrak{R} = \left\{\bar{k} : k \geq 0\right\}$ and consider the operations on $\mathfrak{R}$:

(a) $\bar{k} + \bar{s} = \overline{k+s}$, $\bar{k} \cdot \bar{s} = \overline{k \cdot s}$ $\quad (k > 0, s > 0)$.

(b) $\bar{t} + \bar{0} = \bar{0} + \bar{t} = \bar{t}$, $\bar{t} \cdot \bar{0} = \bar{0} \cdot \bar{t} = \bar{0}$ $\quad (t \geq 0)$.

Then, $(\mathfrak{R}, +, \cdot)$ is an isomorphic structure to the usual one $(\mathbb{R}^+, +, \cdot)$ and prime numbers $p \in \mathbb{N}$ are characterized by the fact that $\bar{p} \neq \bar{1}$ and $\bar{p}$ contains one and only one vortex point.

Amongst the $\mathbb{R}^+$ coding functions that identifies primes, it will be interesting to select those given by $\psi_m : [m, m+1] \to \mathbb{R}^+$ $(m = 0, 1, 2, ...)$ functions that are affine (Fig. 14)

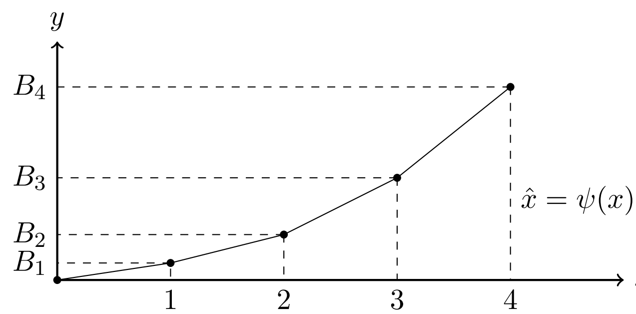


Figure 14: $\mathbb{R}^+$ prime coding

(1.2)

$$\psi_m(x) = \xi_m(x - m) + B_m \ (\xi_m > 0 \ \forall m \in \mathbb{N}, B_0 = 0, B_m = \sum_{j=0}^{m-1} \xi_j \text{ if } m \geq 1).$$

We can easily prove that the ψ functions defined by means of the sequence $(\psi_m)_{m \geq 0}$ are $\mathbb{R}^+$ coding functions. The conditions (1.1) for ψ to identify primes can now thus be expressed:

$$\psi \text{ identifies primes} \Leftrightarrow (\xi_i \neq \xi_{i+1}) \wedge (\xi_i \xi_j \neq \xi_{i+1} \xi_{j+1}).$$

Equivalently, ψ identifies primes $\Leftrightarrow$ $\xi_i \xi_j \neq \xi_{i+1} \xi_{j+1}$ $(\forall i \forall j \in \mathbb{N})$. The fulfilment of this inequality is guaranteed by choosing ξ_i such that $0 < \xi_i < \xi_{i+1}$ $(\forall i \in \mathbb{N})$ though this is not the only way of choosing it.

Definition 1.14. Any $\mathbb{R}^+$ coding function ψ that is defined by means of ψ_m affine functions that also satisfies $0 < \xi_i < \xi_{i+1}$ $(\forall i \in \mathbb{N})$ it is said to be an $\mathbb{R}^+$ *prime coding.* We call the numbers $\xi_0, \xi_1, \xi_2, \xi_3, ...$ coefficients of the $\mathbb{R}^+$ prime coding.

2. Essential regions and Goldbach Conjecture

Goldbach's Conjecture is one of the oldest unsolved problems in number theory and in all of mathematics. It states: *"Every even integer greater than 2 can be written as the sum of two primes"* $(\mathcal{S})$. Furthermore, in his famous speech at the mathematical society of Copenhagen in 1921 G.H. Hardy pronounced that $\mathcal{S}$ is probably " as difficult as any of the unsolved problems in mathematics " and therefore Goldbach problem is not only one of the most famous and difficult problems in number theory, but also in the whole of mathematics ([9]). In this section, and using the Hyperbolic Classification of Natural Numbers we provide a characterization of $\mathcal{S}$.

In the $\hat{x}\hat{y}$ plane determined by any $\mathbb{R}^+$ prime coding function ψ and for any given ψ-even number $\hat{\alpha} \geq \widehat{16}$ we will consider the function in which any number $\hat{k}$ of the closed interval $[\hat{4}, \hat{\alpha} \div \hat{2}]$ corresponds to the area of the region of $\hat{x}\hat{y}$: $\hat{x} \geq \hat{2}, \hat{y} \geq \hat{x}, \hat{x} \otimes \hat{y} \leq \hat{k}$ (called lower area) and also the function that associates each to the area of the region of $\hat{x}\hat{y}$: $\hat{x} \geq \hat{2}, \hat{y} \geq \hat{x}, \hat{\alpha} \sim \hat{k} \leq \hat{x} \otimes \hat{y} \leq \hat{\alpha} \sim \hat{4}$ (called upper area). The $\hat{x}\hat{y}$ plane is considered imbedded in the xy plane with the Lebesgue Measure ([5]). This means that for any given ψ-even number $\hat{\alpha} \geq \widehat{16}$ we have $\hat{\alpha} = \hat{k} \oplus (\hat{\alpha} \sim \hat{k})$ and, associated to this decomposition, two data pieces, lower and upper areas. We will study if $\hat{\alpha}$ the ψ-sum of the two ψ-prime numbers $\hat{k}_0$ and $\hat{\alpha} \sim \hat{k}_0$ taking into account the restrictions $\hat{\alpha} \sim \hat{3}$ and $\hat{\alpha} \div \hat{3}$ both ψ-composite. The upper and lower area functions will not yet yield any characterizations to the Goldbach Conjecture. We will need the second derivative of the total area function (the sum of the lower and upper areas).

To this end, we define the concept of essential regions associated to a hyperbola which, simply put, is any region in the xy plane with the shape $[n, n+1] \times [m, m+1]$ where n and m are natural numbers, $m > n > 1$ and the hyperbola intersects it in more than one point or else the shape $[n, n+1]^2$ where $n > 1$ and $x \leq y$ and the hyperbola intersects in more than one point.

These essential regions are then transported to the $\hat{x}\hat{y}$ plane by means of the $\psi \times \psi$ function, and we will find the total area function adding the areas determined by each hyperbola in the respective essential regions, and the second derivative of this area function in each essential region. After this process we obtain the formula which determines the second derivative function of the total area $\widehat{A}_T$ in each sub-interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$, $k_0 = 4, 5, ..., \alpha/2 - 1$ a derivative which is continuous

$$(\widehat{A}_T)''(\hat{k}) = \frac{x_{k_0}}{\xi_{k_0}^2} \cdot \frac{1}{k} + \frac{y_{k_0}}{\xi_{\alpha-k_0-1}^2} \cdot \frac{1}{\alpha - k} \quad (\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]).$$

Both x_{k_0} and y_{k_0} are numeric values in homogeneous polynomial of degree two obtained from substituting in their variables the ξ_i coefficients of the ψ $\mathbb{R}^+$ prime coding function . We call $P_{k_0} = (x_{k_0}, y_{k_0})$ an essential point. The study of the behaviour of the second derivative in these intervals allows the following characterization of the Goldbach Conjecture for any even number $\alpha \geq 16$ with the restrictions $\alpha - 3$ and $\alpha/2$ composite:

Claim 2.1. $\alpha \geq 16$, an even number, then, α is the sum of two prime numbers k_0 and $\alpha - k_0$ $(5 \leq k_0 < \alpha/2)$ iff the consecutive essential points P_{k_0-1} and P_{k_0} are repeated, that is, if $P_{k_0-1} = P_{k_0}$.

2.1. **Essential regions associated with a hyperbola.**

Definition 2.2. Consider the family of functions

$$\mathcal{H} = \{h_k : [\,2,\ \sqrt{k}\,] \to \mathbb{R},\ h_k(x) = k/x,\ k \geq 4\}$$

whose graphs represent the pieces of the hyperbolas $xy = k$ $(k \geq 4)$ included in the subset of $\mathbb{R}^2$, $S \equiv (x \geq 2) \wedge (x \leq y)$. For n, m natural numbers consider the subsets of $\mathbb{R}^2$:
a) $R_{(n,m)} = [n, n+1] \times [m, m+1] \quad (2 \leq n < m)$
b) $R_{(n,n)} = ([n, n+1] \times [n, n+1]) \cap \left\{(x, y) \in \mathbb{R}^2 : y \geq x\right\}$
Let h_k be an element of $\mathcal{H}$. We say that $R_{(n,m)}$ is a *square essential region* of h_k iff $R_{(n,m)} \cap \Gamma\,(h_k)$ contains more than one point. We say that $R_{(n,n)}$ is a *triangular essential region* of h_k iff $R_{(n,n)} \cap \Gamma\,(h_k)$ contains more than one point.

Example 2.3. The essential regions of the $xy = 17$ hyperbola are $R_{(2,8)}$, $R_{(2,7)}$, $R_{(2,6)}$, $R_{(2,5)}$, $R_{(3,5)}$, $R_{(3,4)}$ and $R_{(4,4)}$ (Fig. 15).

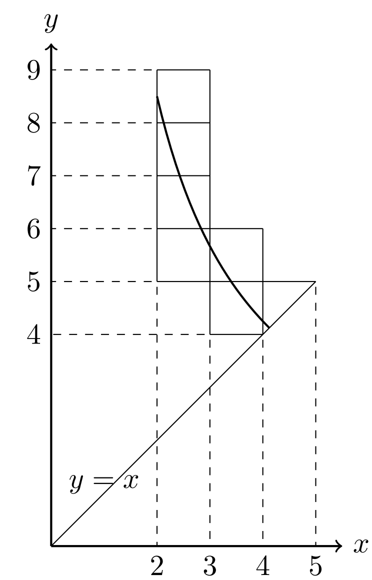


Figure 15: Essential regions of $xy = 17$

Analyse the different types of essential regions depending on the way the hyperbola $xy = k$ intersects with $R_{(n,m)}\,(m > n)$. If the hyperbola passes through point $P\,(n, m+1)$ (Fig.16), then the equation for the hyperbola is $xy = n(m+1)$.

The abscissa of the Q point is $x = n(m+1)/m$. We verify that $n < n(m+1)/m < n+1$. This is equivalent to say $nm < nm+n$ and $nm+n < mn+m$

or equivalently $(0 < n) \wedge (n < m)$, which are trivially true. The remaining types are reasoned in a similar way (Fig. 17).

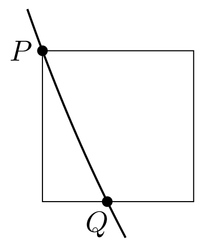


Figure 16: Intersection between hyperbolas and essential regions

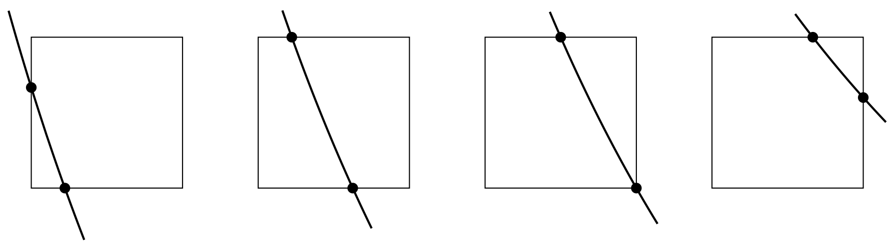


Figure 17: Types of square essential regions

We use the same considerations for the triangular essential regions $R_{(n,n)}$ (Fig. 18)

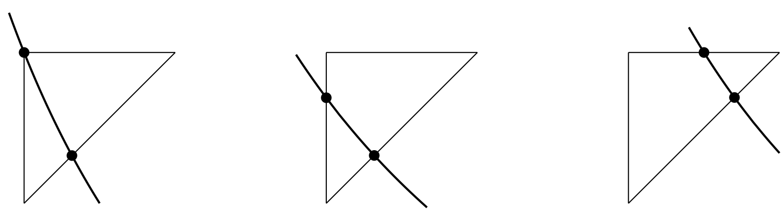


Figure 18: Types of triangular essential regions

Let $k_0 \in \mathbb{N}, k_0 \geq 4$. We will examine which are the types of essential regions for the hyperbolas $xy = k$ $(y \geq x)$ where $k_0 < k < k_0 + 1$. The passage through essential regions of points P_0, Q_0 of the $xy = k_0$ hyperbola with relation to P, Q points of the $xy = k$ hyperbola corresponds to the following diagrams (Fig. 19)

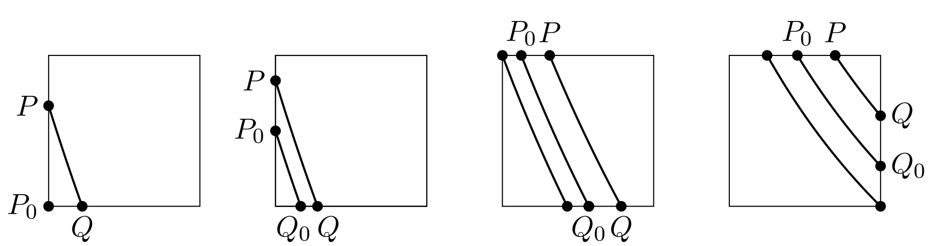


Figure 19: Square essential regions $(k_0 < k < k_0 + 1)$

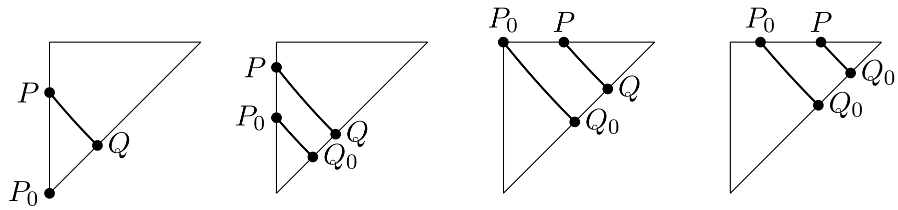


Figure 20: Triangular essential regions ($k_0 < k < k_0 + 1$)

As a consequence, the essential regions for the hyperbola $xy = k$ ($k > 4$) are of the following types

a) Square essential regions $R_{(n,m)}$

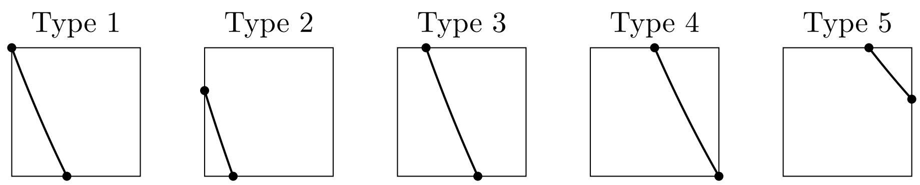


Figure 21: Square essential regions ($k > 4$)

b) Triangular essential regions $R_{(n,n)}$

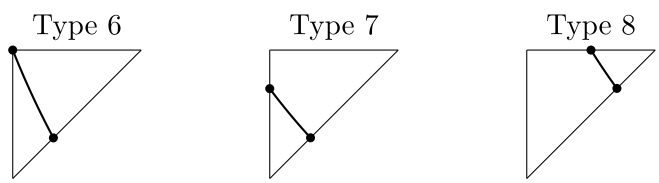


Figure 22: Triangular essential regions ($k > 4$)

We will find the essential regions of the $xy = k$ hyperbolas with the conditions $k_0 \in \mathbb{N}$, $k_0 \geq 4$, $k_0 < k < k_0 + 1$. The abscissa of $xy = k_0$ varies in the interval $[2, \sqrt{k_0}]$

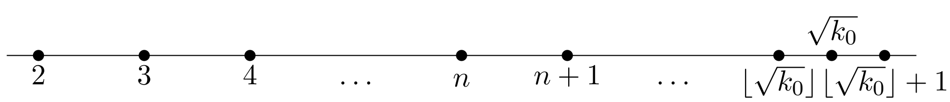


Figure 23: Finding all essential regions (1)

a) For $n \in \{2, 3, \ldots, \lfloor\sqrt{k_0}\rfloor - 1\}$ the $R_{(n,m)}$ essential regions of the $xy = k$ hyperbolas are obtained when m varies in the set (fig. 24):

$$\{\lfloor k_0/(n+1)\rfloor, \lfloor k_0/(n+1)\rfloor + 1, ..., \lfloor k_0/n\rfloor\}.$$

We can easily verify that if $m = \lfloor k_0/n\rfloor$ then $R_{(n,m)}$ is a square essential region of Type 2, if $m = \lfloor k_0/(n+1)\rfloor$, $R_{(n,m)}$ is a square essential region of Type 5 and the remaining $R_{(n,m)}$ are of Type 3 (Fig. 21).

b) For $n = \lfloor\sqrt{k_0}\rfloor$, the $R_{(\lfloor\sqrt{k_0}, m\rfloor)}$ essential regions are obtained when m varies in the set:

$$\{\lfloor\sqrt{k_0}\rfloor, \lfloor\sqrt{k_0}\rfloor + 1, \ldots, \lfloor k_0/\lfloor\sqrt{k_0}\rfloor\rfloor\}.$$

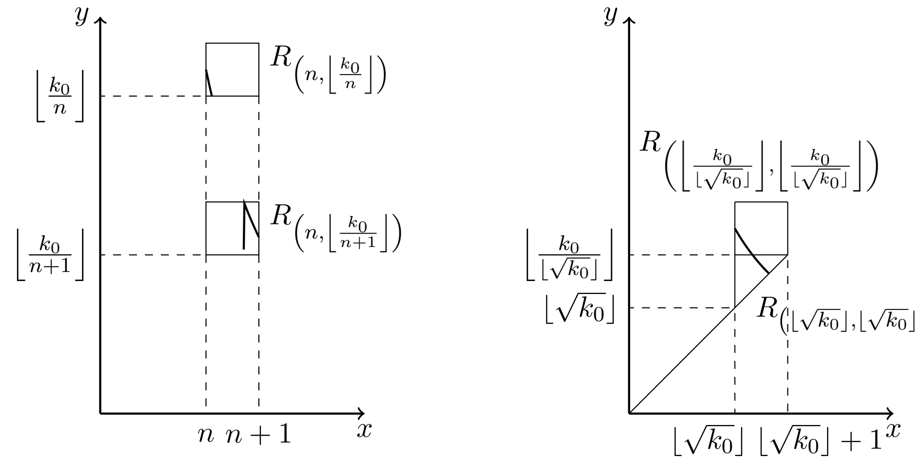


Figure 24: Finding all essential regions (2)

If $m = \lfloor\sqrt{k_0}\rfloor$ we obtain a triangular essential region and could eventually exist a square essential region (Fig. 24). Consider the set of indexes $\{(n, i_n)\}$ such that

(1) For $n = 2, 3, \ldots, \lfloor\sqrt{k_0}\rfloor - 1$ then

$$i_n = \lfloor k_0/(n+1)\rfloor, \lfloor k_0/(n+1)\rfloor + 1, \ldots, \lfloor k_0/n\rfloor.$$

(2) For $n = \lfloor\sqrt{k_0}\rfloor$ then

$$i_n = \lfloor\sqrt{k_0}\rfloor, \lfloor\sqrt{k_0}\rfloor + 1, \ldots, \lfloor\, k_0/\lfloor\sqrt{k_0}\rfloor\,\rfloor.$$

Let $E_s(k_0)$ be the set $\{(n, i_n)\}$, where (n, i_n) are pairs of type (1) or of type (2). We obtain the following theorem:

Theorem 2.4. *Let $k_0 \in \mathbb{N}^*$ ($k_0 \geq 4$). Then,*
i) All the $xy = k$ ($k_0 < k < k_0 + 1$) hyperbolas have the same essential regions, each of the same type.
ii) The $xy = k$ essential regions are the elements of the set

$$\{R_{(n,i_n)} : (n, i_n) \in E_s(k_0)\}.$$

Example 2.5. For $k_0 = 18$ the essential regions of the $xy = k$ ($18 < k < 19$) hyperbolas are (Fig. 25) $R_{(2,9)}$, $R_{(3,6)}$ (type 2), $R_{(2,8)}$, $R_{(2,7)}$, $R_{(3,5)}$ (type 3), $R_{(2,6)}$, $R_{(3,4)}$ (type 5) and $R_{(4,4)}$ (type 7).

The essential regions of the $xy = k$ ($19 < k < 20$) hyperbolas are exactly the same, due to the fact that 19 is a prime number.

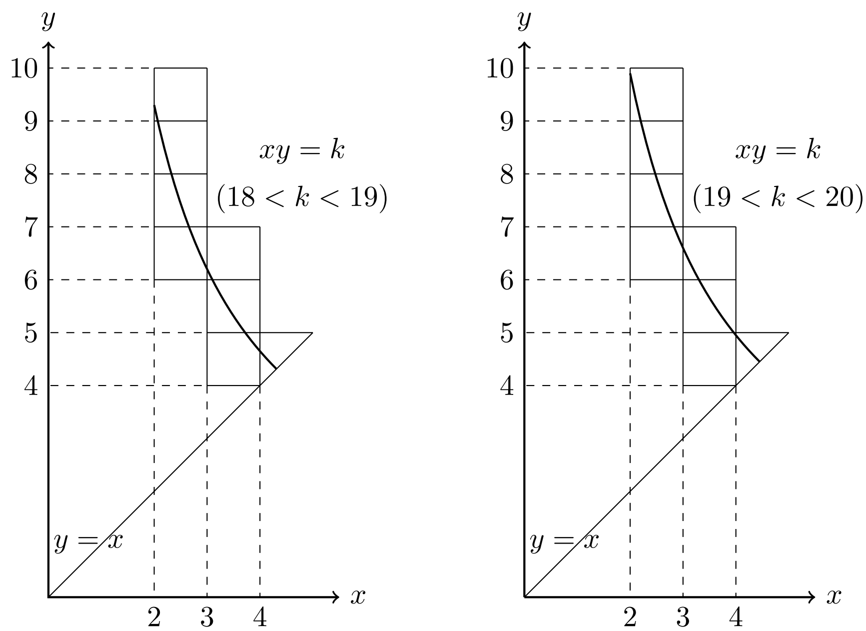


Figure 25: Essential regions ($18 < k < 19$ and $19 < k < 20$)

2.2. **Areas in essential regions associated with a hyperbola.** To every $R_{(n,m)}$ ($n \leq m$) essential region of the $xy = k$ ($k \notin \mathbb{N}^*, k > 4$) hyperbola, we will associate the region of the xy plane below the hyperbola (we call it $D_{(n,m)}(k)$). Denote $A_{(n,m)}(k)$ the area of $D_{(n,m)}(k)$. We have the following cases (Fig. 26).

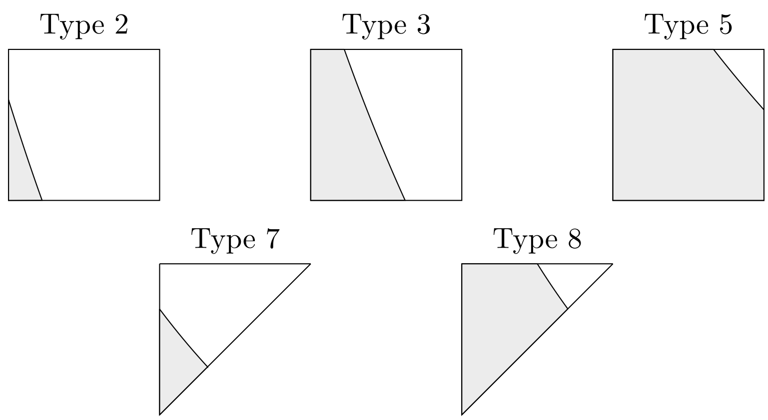


Figure 26: Areas in essential regions

(i) Type 2 essential region

$A_{(n,m)}(k) = \iint_{D_{(n,m)}(k)} dxdy$ with $D_{(n,m)}(k) \equiv n \leq x \leq k/m\ , m \leq y \leq k/x$.

$$\begin{aligned}A_{(n,m)}(k) &= \int_n^{\frac{k}{m}} dx \int_m^{\frac{k}{x}} dy\\ &= \int_n^{\frac{k}{m}} \left(\frac{k}{x} - m\right) dx\\ &= k \log \frac{k}{nm} + nm - k.\end{aligned}$$

If $k \in [k_0, k_0+1]$ ($k_0 \geq 4$ natural number), then $A'_{(n,m)}(k) = \log k/(nm)$ and the second derivative is $A''_{(n,m)}(k) = 1/k$. Note that we have used the closed interval $[k_0, k_0+1]$ so we may extend the definition of the essential region for $k \in \mathbb{N}$ ($k \geq 4$) in a natural manner. In some cases the "essential region" would consist of a single point (null area).

(*ii*) Type 3 essential region

In this case $D_{(n,m)}(k) = D' \cup D''$ where $D' = [n, k/(m+1)] \times [m, m+1]$ and $D'' \equiv k/(m+1) < x \leq k/m$, $m \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$\begin{aligned}A_{(n,m)}(k) &= \iint_{D_{(n,m)}(k)} dxdy\\ &= \frac{k}{m+1} - n + \iint_{D''} dxdy\\ &= \frac{k}{m+1} - n + k \log \frac{m+1}{m} + mk\left(\frac{1}{m+1} - \frac{1}{m}\right).\end{aligned}$$

If $k_0 \leq k \leq k_0 + 1$ then, $A''_{(n,m)}(k) = 0$.

(*iii*) Type 5 essential region

In this case $D_{(n,m)}(k) = D' \cup D''$ where $D' = [n, k/(m+1)] \times [m, m+1]$ and $D'' \equiv k/(m+1) < x \leq n+1$, $m \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$\begin{aligned}A_{(n,m)}(k) &= \iint_{D_{(n,m)}(k)} dxdy\\ &= \frac{k}{m+1} - n + \iint_{D''} dxdy\\ &= \frac{k}{m+1} - n + k \log \frac{(n+1)(m+1)}{k} - m\left(n + 1 - \frac{k}{m+1}\right).\end{aligned}$$

In the interval $[k_0, k_0+1]$ we obtain $A'_{(n,m)}(k) = \log((n+1)(m+1)/k)$ and $A''_{(n,m)}(k) = -1/k$.

(*iv*) Type 7 essential region

$$D_{(n,n)}(k) \equiv n \leq x \leq \sqrt{k}\ ,\ x \leq y \leq k/x.$$

$$\begin{aligned}
A_{(n,n)}(k) &= \int_n^{\sqrt{k}} dx \int_x^{\frac{k}{x}} dy \\
&= \int_n^{\sqrt{k}} \left(\frac{k}{x} - x\right) dx \\
&= \left[k \log x - \frac{x^2}{2}\right]_n^{\sqrt{k}} \\
&= \frac{k}{2}\log k - \frac{k}{2} - k\log n + \frac{n^2}{2}.
\end{aligned}$$

If $k_0 \leq k \leq k_0 + 1$, $A'_{(n,n)}(k) = (1/2)\log k - \log n$ and $A''_{(n,n)}(k) = 1/2k$.

<u>(v) Type 8 essential region</u>

In this case $D_{(n,n)}(k) = D' \cup D''$ where $D' \equiv n \leq x \leq k/(n+1),\ x \leq y \leq n+1$ and $D'' \equiv k/(n+1) < x \leq \sqrt{k},\ x \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$\begin{aligned}
A_{(n,n)}(k) &= \iint_{D'} dxdy + \iint_{D''} dxdy \\
&= \int_n^{\frac{k}{n+1}} dx \int_x^{n+1} dy + \int_{\frac{k}{n+1}}^{\sqrt{k}} dx \int_x^{\frac{k}{x}} dy \\
&= \int_n^{\frac{k}{n+1}} (n+1-x)\, dx + \int_{\frac{k}{n+1}}^{\sqrt{k}} \left(\frac{k}{x} - x\right) dx \\
&= \frac{k}{2} - n(n+1) + \frac{n^2}{2} + k\log\frac{n+1}{\sqrt{k}}.
\end{aligned}$$

If $k_0 \leq k \leq k_0 + 1$, $A'_{(n,n)}(k) = \log((n+1)/\sqrt{k})$ and $A''_{(n,n)}(k) = -1/2k$.

2.3. **Areas of essential regions in the $\hat{x}\hat{y}$ plane.** Consider in the xy plane, an essential region $R_{(n,m)}(n \leq m)$ of the $xy = k$ $(k \geq 4)$ hyperbola and ψ an $\mathbb{R}^+$ prime coding function with ξ_i coefficients. Let $\widehat{R}_{(n,m)}$ be the corresponding region in the $\hat{x}\hat{y}$ plane that is, $\widehat{R}_{(n,m)} = (\psi \times \psi)(R_{(n,m)})$. We call $\widehat{A}_{(n,m)}$ the area of $\widehat{D}_{(n,m)} = (\psi \times \psi)(D_{(n,m)})$ supposing the $\hat{x}\hat{y}$ plane embedded in the xy plane.

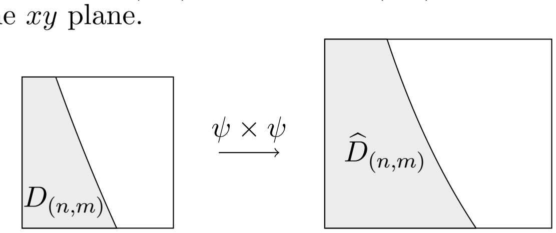


Figure 27: Relationship between $\widehat{A}_{(n,m)}$ and $A_{(n,m)}$

Theorem 2.6. *In accordance with the aforementioned conditions*

$$\widehat{A}_{(n,m)} = \xi_n \xi_m A_{(n,m)}.$$

Proof. The transformation that maps $D_{(n,m)}$ in $\widehat{D}_{(n,m)}$ is $\hat{x} = \psi_n(x),\ \hat{y} = \psi_m(y)$. The Jacobian for this transformation is

$$J = \det \begin{bmatrix} \dfrac{\partial \hat{x}}{\partial x} & \dfrac{\partial \hat{x}}{\partial y} \\ \dfrac{\partial \hat{y}}{\partial x} & \dfrac{\partial \hat{y}}{\partial y} \end{bmatrix} = \det \begin{bmatrix} \psi_n'(x) & 0 \\ 0 & \psi_m'(y) \end{bmatrix} = \psi_n'(x)\psi_m'(y) \neq 0.$$

Thus, ([1]) $\widehat{A}_{(n,m)} = \iint_{\widehat{D}_{(n,m)}} d\hat{x}d\hat{y} = \iint_{D_{(n,m)}} |\psi_n'(x)\psi_m'(y)|\ dxdy$. Since ψ is an $\mathbb{R}^+$ prime coding function, then $|J| = \xi_n\xi_m$ and as a result the relationship between the areas of the essential regions in xy and in $\hat{x}\hat{y}$ is

$$\widehat{A}_{(n,m)} = \iint_{D_{(n,m)}} \xi_n\xi_m\ dxdy = \xi_n\xi_m \iint_{D_{(n,m)}} dxdy = \xi_n\xi_m A_{(n,m)}.$$

□

Let α be an even number. We will assume for technical reasons that $\alpha \geq 16$. Let $k \in [4, \alpha/2]$ and consider the subsets of $\mathbb{R}^2$

$$D_I(k) = \{(x,y) \in \mathbb{R}^2 : x \geq 2,\ y \geq x,\ xy \leq k\},$$
$$D_S(k) = \{(x,y) \in \mathbb{R}^2 : x \geq 2,\ y \geq x,\ \alpha - k \leq xy \leq \alpha - 4\}.$$

Let ψ be an $\mathbb{R}^+$ prime coding function and consider the subsets of $[0, M_\psi)^2$

$$\widehat{D}_I(\hat{k}) = (\psi \times \psi)(D_I(k))\ ,\ \widehat{D}_S(\hat{k}) = (\psi \times \psi)(D_S(k)).$$

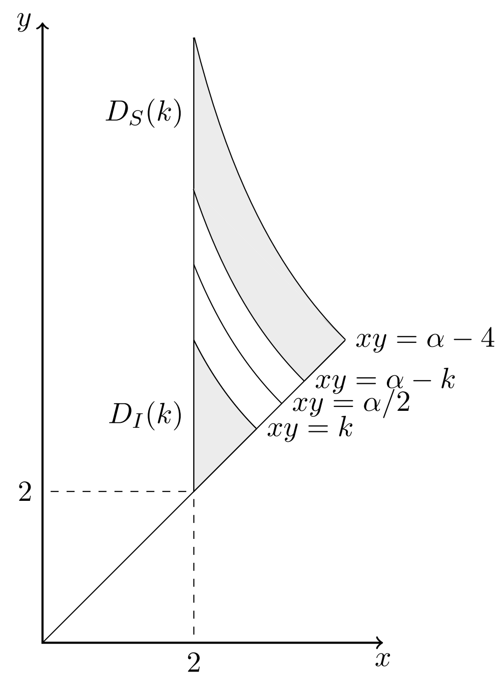


Figure 28: $D_I(k)$ and $D_S(k)$

We now define the functions

1) $\widehat{A}_I : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\hat{k} \to \widehat{A}_I(\hat{k})$ (area of $\widehat{D}_I(\hat{k})$).
2) $\widehat{A}_S : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\hat{k} \to \widehat{A}_S(\hat{k})$ (area of $\widehat{D}_S(\hat{k})$).
3) $\widehat{A}_T : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\widehat{A}_T = \widehat{A}_I + \widehat{A}_S$.

Let α be an even number ($\alpha \geq 16$) and ψ an $\mathbb{R}^+$ prime coding function with coefficients ξ_i. We take $k_0 = 4, 5, \ldots, \alpha/2 - 1$ and we study the second derivative of $\widehat{A}_I$ at each closed interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. For this, we consider the corresponding function $A_I(k)$. Then $\forall k \in [k_0, k_0 + 1]$ we verify $A_I(k) = A_I(k_0) + A_I(k) - A_I(k_0)$. Additionally, $A_I(k) - A_I(k_0)$ is the sum of the areas in the essential regions associated with the $xy = k$ hyperbola, minus the area in the essential regions associated with the $xy = k_0$ hyperbola so,

$$A_I(k) - A_I(k_0) = \sum_{(n,i_n) \in E_S(k_0)} \left[A_{(n,i_n)}(k) - A_{(n,i_n)}(k_0) \right].$$

We know that functions $A_{(n,i_n)}(k)$ have a second derivative in $[k_0, k_0 + 1]$, therefore

$$A_I''(k) = \sum_{(n,i_n) \in E_S(k_0)} A_{(n,i_n)}''(k) \quad (\forall k \in [k_0, k_0 + 1]).$$

We now want to find the expression of $(\widehat{A}_I)''$ as a function of the variable $\hat{k}$, where $\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. By proposition 2.6, $\widehat{A}_{(n,m)}(\hat{k}) = \xi_n \xi_m A_{(n,m)}(k)$. If we derive with respect to $\hat{k}$, we obtain

$$(\widehat{A}_{(n,m)})'(\hat{k}) = \xi_n \xi_m A_{(n,m)}'(k) \frac{dk}{d\hat{k}}.$$

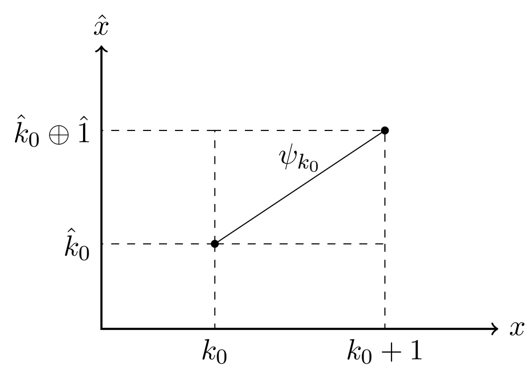


Figure 29: Finding $(\widehat{A}_{(n,m)})''(\hat{k})$ (1)

At $k \in [k_0, k_0 + 1]$, the expression of $\hat{k}$ is $\hat{k} = \xi_{k_0}(k - k_0) + B_{k_0}$ (1.2). Then $dk/d\hat{k} = 1/\xi_{k_0}$, therefore $(\widehat{A}_{(n,m)})'(\hat{k}) = (\xi_n \xi_m / \xi_{k_0}) A_{(n,m)}'(k)$. Deriving once again:

$$(\widehat{A}_{(n,m)})''(\hat{k}) = \frac{\xi_n \xi_m}{\xi_{k_0}^2} A_{(n,m)}''(k).$$

We get the following theorem:

Theorem 2.7. *Let α be an even number ($\alpha \geq 16$). Then for every $\hat{k}_0 = \hat{4}, \hat{5}, \ldots, (\hat{\alpha} \div \hat{2}) \sim \hat{1}$*

a) $(\widehat{A}_I)''(\hat{k}) = \displaystyle\sum_{(n,i_n)\in E_S(k_0)} (\widehat{A}_{(n,i_n)})''(\hat{k}) \quad (\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]).$

b) For $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ and bearing in mind the different types of essential regions, we obtain

(i) Type 2 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = \dfrac{\xi_n \xi_m}{\xi_{k_0}^2} \cdot \dfrac{1}{k}.$

(ii) Type 3 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = 0.$

(iii) Type 5 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = -\dfrac{\xi_n \xi_m}{\xi_{k_0}^2} \cdot \dfrac{1}{k}.$

(iv) Type 7 essential region: $(\widehat{A}_{(n,n)})''(\hat{k}) = \dfrac{\xi_n^2}{\xi_{k_0}^2} \cdot \dfrac{1}{2k}.$

(v) Type 8 essential region: $(\widehat{A}_{(n,n)})''(\hat{k}) = -\dfrac{\xi_n^2}{\xi_{k_0}^2} \cdot \dfrac{1}{2k}.$

Example. We will find $(\widehat{A}_I)''(\hat{k})$ in $[\widehat{12}, \widehat{13}]$ with $\hat{\alpha} \geq \widehat{26}$ (Fig. 31).

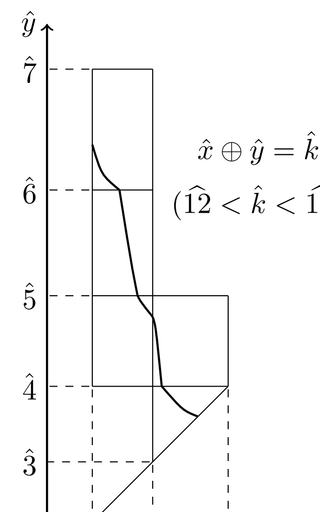


Figure 31: Finding $(\widehat{A}_I)''(\hat{k})$ in $[\widehat{12}, \widehat{13}]$

$$\begin{aligned}(\widehat{A}_I)''(\hat{k}) &= \frac{\xi_2 \xi_6}{\xi_{12}^2} \cdot \frac{1}{k} - \frac{\xi_2 \xi_4}{\xi_{12}^2} \cdot \frac{1}{k} + \frac{\xi_3 \xi_4}{\xi_{12}^2} \cdot \frac{1}{k} - \frac{1}{2} \cdot \frac{\xi_3^2}{\xi_{12}^2} \cdot \frac{1}{k} \\ &= \frac{1}{k\xi_{12}^2}(\xi_2 \xi_6 - \xi_2 \xi_4 + \xi_3 \xi_4 - \xi_3^2/2)\end{aligned}$$

Now, consider the polynomial $p(t) = t_2t_6 - t_2t_4 + t_3t_4 - t_3^2/2$. We call this polynomial a *lower essential polynomial* of $k_0 = 12$ and we write it as P_{I,k_0}.

Definition 2.8. Let α be an even number ($\alpha \geq 16$). The polynomial obtained naturally by removing the common factor function $1/(k\xi_0^2)$ in $(\widehat{A}_I)''(\hat{k})$ in the interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ ($k_0 = 4, 5, \ldots, \alpha/2 - 1$) is called a *lower essential polynomial* of k_0. It is written as P_{I,k_0}.

Remarks 2.9. (i) Lower essential polynomials are homogeneous polynomials of degree 2. (ii) The variables that intervene in P_{I,k_0} are at most t_n and t_{i_n} where $(n, i_n) \in E_s(k_0)$, some of which may be missing (those which correspond to essential regions in which the second derivative is 0). (iii) We will also use P_{I,k_0} as the coefficient of $1/(k\xi_{k_0}^2)$ in $(\widehat{A}_I)''(\hat{k})$.

Corollary 2.10. Let α be an even number ($\alpha \geq 16$). Then, $\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ with $\hat{k}_0 \in \{\hat{4}, \hat{5}, \ldots, \hat{\alpha} \div \hat{2} \sim \hat{1}\}$ we verify $(\widehat{A}_I)''(\hat{k}) = P_{I,k_0}/(k\xi_{k_0}^2)$.

2.4. $(\widehat{A}_S)''$ **and** $(\widehat{A}_T)''$ **functions.** Let α be an even number ($\alpha \geq 16$). We take $k_0 \in \{4, 5, \ldots, \alpha/2 - 1\}$ and we examine the second derivative of $\widehat{A}_S$ at each closed interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. Then, $\forall k \in [k_0, k_0 + 1]$ we verify $A_S(k) = A_S(k_0) + A_S(k) - A_S(k_0)$. Additionally, $A_S(k) - A_S(k_0)$ is the area included between the curves

$$xy = \alpha - k_0,\ xy = \alpha - k,\ x = 2,\ y = x.$$

As a result, it is the sum of the areas in the essential regions of the $xy = \alpha - k_0$ hyperbola minus the area in the essential regions of $xy = \alpha - k$. We obtain:

$$A_S(k) - A_S(k_0) = \sum_{(n,i_n) \in E_S(\alpha - k_0 - 1)} \left[A_{(n,i_n)}(\alpha - k_0) - A_{(n,i_n)}(\alpha - k)\right],$$

$$A_S''(k) = - \sum_{(n.i_n) \in E_S(\alpha - k_0 - 1)} A''_{(n,i_n)}(\alpha - k).$$

Of course, the same relationships as in the lower areas are maintained with the expression $(\widehat{A}_S)''$ as a function of $\hat{k}$. We are left with:

$$(\widehat{A}_{(n,i_n)})''(\hat{k}) = -\frac{\xi_n \xi_{i_n}}{\xi_{\alpha - k_0 - 1}^2} \cdot A''_{(n,i_n)}(\alpha - k).$$

We define upper essential polynomial in a similar way we defined lower essential polynomial and we write them as P_{S,k_0}. The same remarks are maintained.

Remarks 2.11. (i) Upper essential polynomials are homogeneous polynomials of degree 2. (ii) The variables that intervene in P_{S,k_0} are at most t_n, t_{i_n} where $(n, i_n) \in E_S(\alpha - k_0 - 1)$, some of which may be missing (those which correspond to essential regions in which the second derivative is 0). (iii) We will also use P_{S,k_0} as the coefficient of $1/(\alpha - k)\xi_{\alpha - k_0 - 1}^2$ in $(\widehat{A}_S)''(\hat{k})$.

2.5. **Signs of the essential point coordinates.**

Definition 2.12. Let ψ be an $\mathbb{R}^+$ prime coding function and α an even number ($\alpha \geq 16$). For $k_0 \in \{4, 5, \dots, \alpha/2 - 1\}$ we write $P_{k_0} = (x_{k_0}, y_{k_0}) = (P_{I,k_0}, P_{S,k_0})$. We call any P_{k_0} an *essential point* associated with ψ.

Hence, we can express

$$(\widehat{A}_T)''(\hat{k}) = \frac{x_{k_0}}{\xi_{k_0}^2} \cdot \frac{1}{k} + \frac{y_{k_0}}{\xi_{\alpha-k_0-1}^2} \cdot \frac{1}{\alpha - k} \quad (\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]). \tag{2.1}$$

The formula from proposition 2.7 is

$$(\widehat{A}_I)''(\hat{k}) = \sum_{(n.i_n) \in E_S(k_0)} (\widehat{A}_{(n,i_n)})''(\hat{k}) \quad (\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]).$$

where the $E_S\,(k_0)$ sub-indexes are:
For $n = 2, 3, \dots\, , \lfloor \sqrt{k_0} \rfloor - 1$,

$$i_n = \lfloor k_0/(n+1) \rfloor, \lfloor k_0/(n+1) \rfloor + 1, \dots\, , \lfloor k_0/n \rfloor. \tag{2.2}$$

For $n = \lfloor \sqrt{k_0} \rfloor$,

$$i_n = \lfloor \sqrt{k_0} \rfloor, \lfloor \sqrt{k_0} \rfloor + 1, \dots\, , \lfloor\, k_0/\lfloor \sqrt{k_0} \rfloor\, \rfloor. \tag{2.3}$$

Thus, for sub-index n in (1), in $(\widehat{A}_I)''$ only intervene $i_n = \lfloor k_0/(n+1) \rfloor$ and $i_n = \lfloor k_0/n \rfloor$, since we have already seen that all the sub-indexes included between them two, $(\widehat{A}_{(n,i_n)})''(\hat{k}) = 0$, as the essential regions are of type 3. In the lower essential polynomial we obtain $\xi_n(\xi_{\lfloor k_0/n \rfloor} - \xi_{[k_0/(n+1)]}) > 0$ (for any $\mathbb{R}^+$ prime coding function). For $n = \lfloor \sqrt{k_0}\ \rfloor$ we obtain the cases:

$$(i)\ \lfloor \sqrt{k_0} \rfloor = \lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor \quad (ii)\ \lfloor \sqrt{k_0} \rfloor < \lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor. \tag{2.4}$$

In case (i) we would obtain the addend $(1/2)\xi^2_{\lfloor\ k_0\ \rfloor}$, in case (ii) we would obtain (Fig. 32):

$$\xi_{\lfloor \sqrt{k_0} \rfloor} \xi_{\lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor} - (1/2)\xi^2_{\lfloor \sqrt{k_0} \rfloor} = \xi_{\lfloor \sqrt{k_0} \rfloor}(\xi_{\lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor} - (1/2)\xi_{\lfloor \sqrt{k_0} \rfloor}) > 0.$$

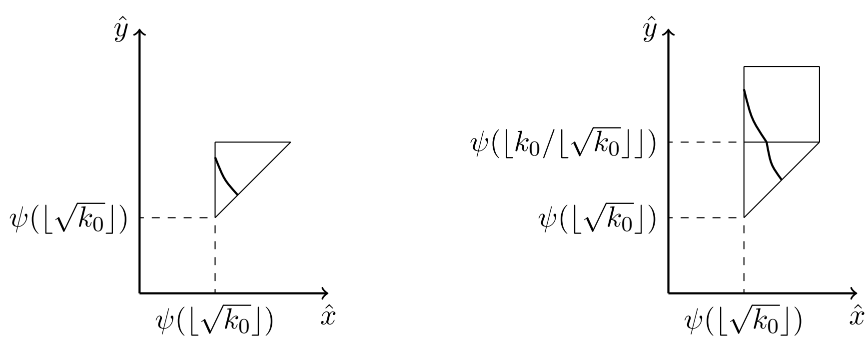
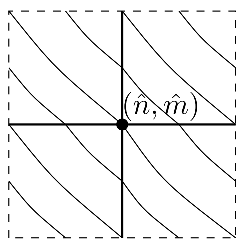


Figure 32: Finding the sign of x_{k_0}

As a result, for an $\mathbb{R}^+$ prime coding function we obtain $x_4 > 0$, $x_5 > 0$, $\ldots$, $x_{\alpha/2-1} > 0$. The reasoning is entirely analogous for the upper essential polynomials that is, $y_4 < 0$, $y_5 < 0$, $\ldots$, $y_{\alpha/2-1} < 0$. We will now arrange the coordinates for the essential points.

1. Lower essential polynomials If $k_0 \in \mathbb{N}$, $(k_0 > 4)$ is composite, there is at least one ψ- natural number coordinates point $(\hat{n}, \hat{m})$ such that $\hat{2} \le \hat{n} \le \hat{m}$ which the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}_0$ goes through.

(a) If $2 < n < m$ we obtain the changes

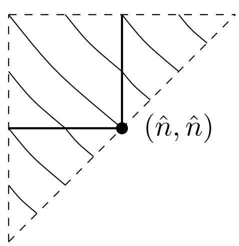


Changes

P_{I,k_0-1}	P_{I,k_0}
0	$-\xi_{n-1}\xi_m$
$-\xi_{n-1}\xi_{m-1}$	$\xi_n\xi_m$
$\xi_n\xi_{m-1}$	0

Figure 33: Arranging x_{k_0} in order (Case a)

(b) If $2 < n = m$

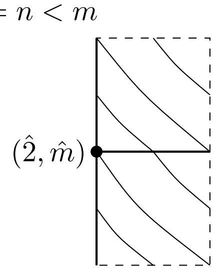


Changes

P_{I,k_0-1}	P_{I,k_0}
0	$-\xi_{n-1}\xi_n$
$-\xi_{n-1}^2/2$	$\xi_n^2/2$

Figure 34: Arranging x_{k_0} in order (Case b)

(c) If $2 = n < m$



Changes

P_{I,k_0-1}	P_{I,k_0}
0	$\xi_2\xi_m$
$\xi_2\xi_{m-1}$	0

Figure 35: Arranging x_{k_0} in order (Case c)

Then $P_{I,k_0} - P_{I,k_0-1} > 0$, since where there are transformations we obtain, for any prime coding function, either (a) or (b) or (c)

$$(a) \quad \begin{aligned} \xi_n\xi_m - \xi_{n-1}\xi_m + \xi_{n-1}\xi_{m-1} - \xi_n\xi_{m-1} &= \\ \xi_m\left(\xi_n - \xi_{n-1}\right) - \xi_{m-1}\left(\xi_n - \xi_{n-1}\right) &= \\ \left(\xi_n - \xi_{n-1}\right)\left(\xi_m - \xi_{m-1}\right) &> 0. \end{aligned}$$

$$(b)\quad \frac{\xi_n^2}{2} - \xi_{n-1}\xi_n + \frac{\xi_{n-1}^2}{2} = \frac{\xi_n^2 - 2\xi_{n-1}\xi_n + \xi_{n-1}^2}{2} = \frac{(\xi_n - \xi_{n-1})^2}{2} > 0.$$

$$(c)\quad \xi_2\xi_m - \xi_2\xi_{m-1} = \xi_2\left(\xi_m - \xi_{m-1}\right) > 0.$$

If k_0 is prime then $P_{I,k_0-1} = P_{I,k_0}$ since the same essential regions exist for the hyperbolas $\hat{x} \otimes \hat{y} = \hat{k}$ in $(\hat{k}_0 \sim \hat{1}, \hat{k}_0) \cup (\hat{k}_0, \hat{k}_0 \oplus \hat{1})$.

2. Upper essential polynomials If $\alpha - k_0$ is composite, and reasoning in the same way, we obtain $P_{S,k_0} - P_{S,k_0-1} > 0$. For $\alpha - k_0$ prime we obtain $P_{S,k_0} = P_{S,k_0-1}$ since the same essential regions exist for the hyperbolas $\hat{x} \otimes \hat{y} = \hat{\alpha} \sim \hat{k}$ if $\hat{k} \in (\hat{k}_0 \sim \hat{1}, \hat{k}_0) \cup (\hat{k}_0, \hat{k}_0 \oplus \hat{1})$. We obtain the theorem:

Theorem 2.13. *Let α be an even number $(\alpha \geq 16)$, and ψ an $\mathbb{R}^+$ prime coding function. Let $P_{k_0} = (x_{k_0}, y_{k_0})$ be the essential points. Then,*
(i) $0 < x_4 \leq x_5 \leq \ldots \leq x_{\alpha/2-1}$. Additionally, $x_{k_0-1} = x_{k_0} \Leftrightarrow k_0$ is prime.
(ii) $y_4 \leq y_5 \leq \ldots \leq y_{\alpha/2-1} < 0$. Additionally, $y_{k_0-1} = y_{k_0} \Leftrightarrow \alpha - k_0$ is prime.

The following Corollary proves claim 2.1 i.e.:

Corollary 2.14. In the hypotheses from the above theorem: The even number α is the sum of two primes k_0 and $\alpha - k_0$, $k_0 \in \{5, 6, \ldots, \alpha/2-1\}$ iff the consecutive essential points P_{k_0-1} and P_{k_0} are repeated, that is $P_{k_0-1} = P_{k_0}$.

3. Time and Arithmetic

> *I have sometimes thought that the profound mystery which envelops our conceptions relative to prime numbers depends upon the limitations of our faculties in regard to time which, like space may be in essence poly-dimensional and that this and other such sort of truths would become self-evident to a being whose mode of perception is according to superficially as opposed to our own limitation to linearly extended time.* (J.J. Sylvester [7])

3.1. **Construction of the Goldbach Conjecture function.**

Theorem 3.1. *Let α be an even number $(\alpha \geq 16)$, and ψ be an $\mathbb{R}^+$ prime coding function with ξ_i coefficients. Let $P_{k_0} = (x_{k_0}, y_{k_0})$ be the essential points $(k_0 = 4, 5, \ldots, \alpha/2-1)$. Then,*
(i) x_{k_0} depends at most on $\xi_2, \xi_3, \ldots, \xi_{\lfloor k_0/2 \rfloor}$.
(ii) y_{k_0} depends at most on $\xi_2, \xi_3, \ldots, \xi_{\lfloor (\alpha-k_0-1)/2 \rfloor}$.

Proof. (i) In $E_S(k_0) = \{(n, i_n)\}$ we verify that $n \leq i_n$. The smallest n is 2 and the biggest i_n is $\lfloor k_0/2 \rfloor$ (ii) In $E_S(\alpha - k_0 - 1) = \{(n, i_n)\}$ we verify that $n \leq i_n$. The smallest n is 2 and the biggest i_n is $\lfloor (\alpha - k_0 - 1)/2 \rfloor$ $\square$

Remarks 3.2. Since the biggest sub-index coefficient that appears at the essential point coordinates is $\xi_{\lfloor(\alpha-4-1)/2\rfloor} = \xi_{\alpha/2-3}$ we conclude that knowing the coefficients $\xi_2, \xi_3, \ldots, \xi_{\alpha/2-3}$ all the essential points are determined. Note that where $0 < \xi_2 < \xi_3 < \ldots < \xi_{\alpha/2-3}$ the 2.14 corollary is met.

This leads to the following definition:

Definition 3.3. Let α be an even number ($\alpha \geq 16$), and ψ an $\mathbb{R}^+$ coding function such that its ξ_i coefficients verify $0 < \xi_2 < \xi_3 < \ldots < \xi_{\alpha/2-1}$ and $\xi_i > 0$ in other case. We say that ψ is an $\mathbb{R}^+$ *coding function adapted to* α. (We have also included $\xi_{\alpha/2-2}$ and $\xi_{\alpha/2-1}$ for technical reasons)

Generally $(\widehat{A}_T)''(\hat{k}_0-) \neq (\widehat{A}_T)''(\hat{k}_0+)$ (Fig. 36). The following proposition provides sufficient conditions for the $(\widehat{A}_T)''$ function to be well defined and continuous in the $[\hat{4}, \hat{\alpha} \div \hat{2}]$ closed interval.

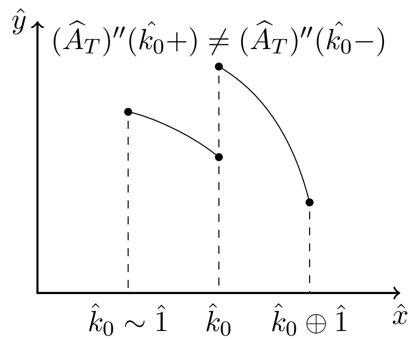


Figure 36: Graph of $(\widehat{A}_T)''$

Theorem 3.4. *Let α be an even number ($\alpha \geq 16$) and ψ be an $\mathbb{R}^+$ coding function adapted to α . Assume that*
i) $\xi_{k_0}^2 x_{k_0-1} = \xi_{k_0-1}^2 x_{k_0}$ *and* $\xi_{\alpha-k_0-1}^2 y_{k_0-1} = \xi_{\alpha-k_0}^2 y_{k_0}$ *for every k_0 composite* $(5 \leq k_0 \leq \alpha/2-1)$
ii) For every p_0 prime $(5 \leq p_0 \leq \alpha/2-1)$,

$$\xi_{\alpha-p_0}^2 = |y_{p_0-1}| \left(\frac{|y_{p_0}|}{\xi_{\alpha-p_0-1}^2} + \frac{\alpha - p_0}{p_0} \cdot x_{p_0-1} \cdot \left(\frac{1}{\xi_{p_0-1}^2} - \frac{1}{\xi_{p_0}^2} \right) \right)^{-1}.$$

Then, $(\widehat{A}_T)''(\hat{k}_0-) = (\widehat{A}_T)''(\hat{k}_0+)$ $\forall k_0 \in \{5, 6, \ldots, \alpha/2-1\}$.

Proof. From (2.1) and for all $k_0 \in \{5, 6, \ldots, \alpha/2-1\}$ we have

$$(\widehat{A}_T)''(\hat{k}_0-) = \frac{x_{k_0-1}}{k_0 \xi_{k_0-1}^2} + \frac{y_{k_0-1}}{(\alpha - k_0)\xi_{\alpha-k_0}^2},$$

$$(\widehat{A}_T)''(\hat{k}_0+) = \frac{x_{k_0}}{k_0 \xi_{k_0}^2} + \frac{y_{k_0}}{(\alpha - k_0)\xi_{\alpha-k_0-1}^2}.$$

Then $(\widehat{A}_T)''(\hat{k}_0-) = (\widehat{A}_T)''(\hat{k}_0+)$ if and only if

$$\frac{1}{k_0}\left(\frac{x_{k_0-1}}{\xi^2_{k_0-1}}-\frac{x_{k_0}}{\xi^2_{k_0}}\right)=\frac{1}{\alpha-k_0}\left(\frac{|y_{k_0-1}|}{\xi^2_{\alpha-k_0}}-\frac{|y_{k_0}|}{\xi^2_{\alpha-k_0-1}}\right).$$

When k_0 is composite, i) implies the equality above. Note that if k_0 is composite, then $x_{k_0-1} < x_{k_0}$, and consequently, $\xi^2_{k_0-1} < \xi^2_{k_0}$, in other words, it is consistent with the hypothesis that ψ is an $\mathbb{R}^+$ coding function adapted to α. If p_0 is prime, then $x_{p_0-1} = x_{p_0}$ therefore $(\widehat{A}_T)''(\hat{p}_0-) = (\widehat{A}_T)''(\hat{p}_0+)$ is equivalent to

$$\frac{\alpha-p_0}{p_0}\cdot x_{p_0-1}\cdot\left(\frac{1}{\xi^2_{p_0-1}}-\frac{1}{\xi^2_{p_0}}\right)=\frac{|y_{p_0-1}|}{\xi^2_{\alpha-p_0}}-\frac{|y_{p_0}|}{\xi^2_{\alpha-p_0-1}},$$

which in turn is equivalent to ii).

□

Now, let α be an even number ($\alpha \geq 16$). We will construct an $\mathbb{R}^+$ coding function adapted to α in such a way that $(\widehat{A}_T)''(\hat{k}_0-) = (\widehat{A}_T)''(\hat{k}_0+)$ for every $k_0 \in \{5, 6, \ldots, \alpha/2-1\}$. We would then have constructed the continuous function

$$\mathfrak{G} : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+,\ \ \mathfrak{G}(\hat{k}) = (\widehat{A}_T)''(\hat{k}).$$

For this we select, at random, $0 < \xi_2 < \xi_3 < \xi_4 < \xi_5$. According to proposition 3.1 $x_4, x_5, \ldots, x_{11}$ are readily determined. We select $\xi^2_6 = (x_6/x_5)\xi^2_5$, then $\xi_6 > \xi_5$, and x_{12} and x_{13} are readily determined. We select $\xi_7 > \xi_6$ at random, and x_{14} and x_{15} are readily determined. We now take $\xi^2_8 = (x_8/x_7)\xi^2_7$, $\xi^2_9 = (x_9/x_8)\xi^2_8$, $\xi^2_{10} = (x_{10}/x_9)\xi^2_9$, then $\xi_7 < \xi_8 < \xi_9 < \xi_{10}$, and $x_{16}, \ldots, x_{21}$ are readily determined. We select $\xi_{11} > \xi_{10}$ at random, and x_{22} and x_{23} are readily determined. Note that for a prime i we are selecting ξ_i at random with the sole condition $\xi_i > \xi_{i-1}$.

Let s_0 be the largest prime such that $s_0 \leq \alpha/2-1$. Then, following the same principle, we take $\xi_{s_0} > \xi_{s_0-1}$ at random, and x_{2s_0} and x_{2s_0+1} are readily determined. Finally, we select

i) If $s_0 \leq \alpha/2-2$

$$\xi^2_{s_0+1}=\frac{x_{s_0+1}}{x_{s_0}}\xi^2_{s_0},\ \xi^2_{s_0+2}=\frac{x_{s_0+2}}{x_{s_0+1}}\xi^2_{s_0+1},\ldots,\xi^2_{\alpha/2-1}=\frac{x_{\alpha/2-1}}{x_{\alpha/2-2}}\xi^2_{\alpha/2-2}.$$

ii) At random $\xi_{\alpha/2-1} > \xi_{\alpha/2-2}$ if $s_0 = \alpha/2-1$.

Following the remarks of proposition 3.1 all the essential points P_{k_0} associated with the number α have been determined. We select $\xi^2_{\alpha/2}$ at random and are only have to determine which are to be the remaining coefficients.

i) If $s_0 = \alpha/2-1$ we select

$$\xi^2_{\alpha/2+1} = \xi^2_{\alpha-(\alpha/2-1)} = \xi^2_{\alpha-s_0} =$$
$$|y_{s_0-1}| \left(\frac{|y_{s_0}|}{\xi^2_{\alpha-s_0-1}} + \frac{\alpha - s_0}{s_0} \cdot x_{s_0-1} \cdot \left(\frac{1}{\xi^2_{s_0-1}} - \frac{1}{\xi^2_{s_0}} \right) \right)^{-1}.$$

ii) If $s_0 < \alpha/2 - 1$ we select

$$\begin{aligned} \xi^2_{\alpha/2+1} &= \xi^2_{\alpha-(\alpha/2-1)} \\ &= |y_{\alpha/2-2}|\, |y_{\alpha/2-1}|^{-1} \xi^2_{\alpha/2}, \\ \xi^2_{\alpha/2+2} &= \xi^2_{\alpha-(\alpha/2-2)} \\ &= |y_{\alpha/2-3}|\, |y_{\alpha/2-2}|^{-1} \xi^2_{\alpha/2+1}, \\ &\cdots \\ \xi^2_{\alpha-s_0-2} &= |y_{s_0+1}|\, |y_{s_0+2}|^{-1} \xi^2_{\alpha/2-s_0-3}, \\ \xi^2_{\alpha-s_0-1} &= |y_{s_0}|\, |y_{s_0+1}|^{-1} \xi^2_{\alpha/2-s_0-2}. \end{aligned}$$

We also verify $\xi^2_{\alpha-s_0-1} = |y_{s_0}|\, |y_{\alpha/2-1}|^{-1} \xi^2_{\alpha/2}$. We now take

$$\xi^2_{\alpha-s_0} = |y_{s_0-1}| \left(\frac{|y_{s_0}|}{\xi^2_{\alpha-s_0-1}} + \frac{\alpha - s_0}{s_0} \cdot x_{s_0-1} \cdot \left(\frac{1}{\xi^2_{s_0-1}} - \frac{1}{\xi^2_{s_0}} \right) \right)^{-1}.$$

Having selected these first coefficients, we construct the remaining coefficients in the following way: for each prime r_0 where $5 \le r_0 < s_0$ we select

$$\xi^2_{\alpha-r_0} = |y_{r_0-1}| \left(\frac{|y_{r_0}|}{\xi^2_{\alpha-r_0-1}} + \frac{\alpha - r_0}{r_0} \cdot x_{r_0-1} \cdot \left(\frac{1}{\xi^2_{r_0-1}} - \frac{1}{\xi^2_{r_0}} \right) \right)^{-1}.$$

Between two consecutive primes p_0 and q_0, such that $5 \le p_0 < q_0 \le s_0$, we select

$$\begin{aligned} \xi^2_{\alpha-q_0+1} &= |y_{q_0-2}||y_{q_0-1}|^{-1} \xi^2_{\alpha-q_0}, \\ \xi^2_{\alpha-q_0+2} &= |y_{q_0-3}||y_{q_0-2}|^{-1} \xi^2_{\alpha-q_0+1}, \\ &\cdots \\ \xi^2_{\alpha-p_0-2} &= |y_{p_0+1}||y_{p_0+2}|^{-1} \xi^2_{\alpha-p_0-3}, \\ \xi^2_{\alpha-p_0-1} &= |y_{p_0}||y_{p_0+1}|^{-1} \xi^2_{\alpha-p_0-2}. \end{aligned}$$

We also verify $\xi^2_{\alpha-p_0-1} = |y_{p_0}||y_{q_0-1}|^{-1} \xi^2_{\alpha-q_0}$. We have now chosen the coefficients $\xi_2,\ \xi_3,\ \ldots,\ \xi_{\alpha/2-1},\ \xi_{\alpha/2},\ \xi_{\alpha/2+1},\ \ldots,\ \xi_{\alpha-5}$. The remaining coefficients of the $\mathbb{R}^+$ coding function adapted to α are irrelevant. Due to the actual construction of these coefficients, the hypotheses in theorem 3.4 are verified, and we have therefore constructed the following continuous function

$$\mathfrak{G} : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+, \quad \mathfrak{G}(\hat{k}) = (\widehat{A}_T)''(\hat{k}).$$

Definition 3.5. We call *Goldbach Conjecture function associated to* α any function $\mathfrak{G}$ constructed in this manner.

Theorem 3.6. *Let* $\mathfrak{G}$ *be a Goldbach Conjecture function with coefficients* ξ_i *associated to* α. *Let* $\mathfrak{P} = \{r_0 : r_0 \text{ prime},\ 5 \le r_0 \le \alpha/2 - 1\}$ *and let* s_0 *be the maximum of* $\mathfrak{P}$. *We call*

$$F_{r_0} = \frac{\alpha - r_0}{r_0} \cdot x_{r_0-1} \cdot \left(\frac{1}{\xi^2_{r_0-1}} - \frac{1}{\xi^2_{r_0}} \right).$$

Then, $\xi^2_{\alpha-5} = |y_4| \; (\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + \sum\limits_{r_0 \in \mathfrak{P}} F_{r_0} \;)^{-1}$.

Proof. According to the construction of any Goldbach Conjecture function $\mathfrak{G}$ we verify $\xi^2_{\alpha-s_0} = |y_{s_0-1}| \; (\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + F_{s_0} \;)^{-1}$ regardless of the fact that $s_0 = \alpha/2 - 1$ or $s_0 < \alpha/2 - 1$. We now define the function $\gamma : \mathfrak{P} - \{5\} \to \mathfrak{P} - \{s_0\}$, $\gamma\,(p)$ as the prime number before p. Let $q_0 \in \mathfrak{P} - \{5\}$ and assume that

$$\xi^2_{\alpha-q_0} = |y_{q_0-1}| \; (\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + F_{s_0} + F_{\gamma(s_0)} + F_{\gamma^2(s_0)} + ... + F_{\gamma^h(s_0)} \;)^{-1}$$

where $\gamma^h(s_0) = q_0$. Now, let $\gamma\,(q_0) = p_0$. Thus, due to the construction of the $\mathfrak{G}$ function we verify

$$\begin{aligned}
\xi^2_{\alpha-p_0} &= |y_{p_0-1}| \; (\; |y_{p_0}| \xi^{-2}_{\alpha-p_0-1} + F_{p_0} \;)^{-1} \\
&= |y_{p_0-1}| \; (\; |y_{q_0-1}| \xi^{-2}_{\alpha-q_0} + F_{p_0} \;)^{-1} \\
&= |y_{p_0-1} \; |(\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + F_{s_0} + F_{\gamma(s_0)} + \ldots + F_{\gamma^h(s_0)} + F_{\gamma^{h+1}(s_0)} \;)^{-1}.
\end{aligned}$$

As a consequence, and taking $p_0 = 5$, we obtain

$$\begin{aligned}
\xi^2_{\alpha-5} &= |y_4| \; (\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + F_5 + F_7 + F_{11} + ... + F_{s_0} \;)^{-1} \\
&= |y_4| \; (\; |y_{\alpha/2-1}| \xi^{-2}_{\alpha/2} + \sum_{r_0 \, \in \, \mathfrak{P}} F_{r_0} \;)^{-1}.
\end{aligned}$$

□

Let α be an even number ($\alpha \geq 16$). Let $\mathfrak{G}$ be any Goldbach Conjecture function associated to α. Then, the $\mathfrak{G}$ coefficients can be expressed in the following way, where $\lambda_i \in (1, +\infty)$ for every $i \in \mathcal{J} = \{3, 4\} \cup \mathfrak{P}$:

$$\xi_2 > 0, \; \xi_3 = \lambda_3 \xi_2, \; \xi_4 = \lambda_4 \xi_3, \; \xi_5 = \lambda_5 \xi_4, \; \xi_{p_0} = \lambda_{p_0} \xi_{p_0-1} \; (\forall p_0 \in \mathfrak{P} - \{5\}).$$

According to the construction of $\mathfrak{G}$, all the coefficients depend exclusively on the variables ξ_2, λ_i and $\xi_{\alpha/2} > 0$. We denote $\bar{\lambda} = (\lambda_i)$ ($i \in \mathcal{J}$) thus, any Goldbach Conjecture function associated to α can be written

$$\mathfrak{G} = \mathfrak{G}_{(\alpha, \, \xi_2, \, \xi_{\alpha/2}, \, \bar{\lambda})} \quad (\alpha \geq 16, \; \xi_2 > 0, \; \xi_{\alpha/2} > 0, \; \lambda_i > 1).$$

Theorem 3.7. *Let* $\mathfrak{G} = \mathfrak{G}_{(\alpha, \, \xi_2, \, \xi_{\alpha/2}, \, \bar{\lambda})}$ *be a Goldbach Conjecture function for the even number* α ($\alpha \geq 16$) *with coefficients* ξ_i ($2 \leq i \leq \alpha - 5$). *Let us denote for every* $p_0 \in \mathfrak{P} - \{5\}$, $P(p_0) := \{p : \; p \text{ prime}, \; 5 \leq p \leq \gamma(p_0)\}$. *Then,* $\forall p_0 \in \mathfrak{P} - \{5\}$ *we verify*

(*a*) $\quad \dfrac{x_{p_0}}{\xi^2_{p_0-1}} = \dfrac{1}{2\lambda_3^2 \lambda_4^2} \displaystyle\prod_{j \in P(p_0)} \dfrac{1}{\lambda_j^2}.$

(b) $\quad F_5 = \dfrac{\alpha - 5}{5} \cdot \dfrac{1}{2\lambda_3^2\lambda_4^2}\left(1 - \dfrac{1}{\lambda_5^2}\right).$

(c) $\quad F_{p_0} = \dfrac{\alpha - p_0}{p_0} \cdot \dfrac{1}{2\lambda_3^2\lambda_4^2}\prod_{j\in P(p_0)}\dfrac{1}{\lambda_j^2}\left(1 - \dfrac{1}{\lambda_{p_0}^2}\right).$

Proof. (a) The equality is true when $p_0 = 7$. In fact, according to the construction of $\mathfrak{G}$, we have

$$\frac{x_7}{\xi_6^2} = \frac{x_6}{\xi_6^2} = \frac{x_5}{\xi_5^2} = \frac{x_4}{\lambda_5^2\xi_4^2} = \frac{\xi_2^2}{2\lambda_5^2\lambda_4^2\lambda_3^2\xi_2^2} = \frac{1}{2\lambda_3^2\lambda_4^2}\prod_{j\in P(7)}\frac{1}{\lambda_j^2}.$$

We have used the fact that $x_4 = P_{I,4} = \xi_2^2/2$. Assume that the equality is true for a prime $p_0 \in \mathfrak{P} - \{5, s_0\}$, we prove that it is also true for the next prime q_0 . With the actual construction of $\mathfrak{G}$, we obtain

$$\xi_{q_0-1}^2 = \frac{x_{q_0-1}}{x_{p_0}}\xi_{p_0}^2 = \frac{x_{q_0}}{x_{p_0}}\xi_{p_0}^2 \Rightarrow \frac{x_{q_0}}{\xi_{q_0-1}^2} = \frac{x_{p_0}}{\xi_{p_0}^2} = \frac{x_{p_0}}{\lambda_{p_0}^2\xi_{p_0-1}^2}$$
$$= \frac{1}{\lambda_{p_0}^2}\cdot\frac{1}{2\lambda_3^2\lambda_4^2}\prod_{j\in P(p_0)}\frac{1}{\lambda_j^2} = \frac{1}{2\lambda_3^2\lambda_4^2}\prod_{j\in P(q_0)}\frac{1}{\lambda_j^2}.$$

(b)

$$F_5 = \frac{\alpha - 5}{5}x_4\left(\frac{1}{\xi_4^2} - \frac{1}{\xi_5^2}\right) = \frac{\alpha - 5}{5}\cdot\frac{x_4}{\xi_4^2}\left(1 - \frac{1}{\lambda_5^2}\right)$$
$$= \frac{\alpha - 5}{5}\cdot\frac{1}{2\lambda_3^2\lambda_4^2}\left(1 - \frac{1}{\lambda_5^2}\right).$$

(c) For all $p_0 \in \mathfrak{P} - \{5\}$

$$F_{p_0} = \frac{\alpha - p_0}{p_0}\cdot x_{p_0-1}\left(\frac{1}{\xi_{p_0-1}^2} - \frac{1}{\xi_{p_0}^2}\right)$$
$$= \frac{\alpha - p_0}{p_0}\cdot\frac{x_{p_0}}{\xi_{p_0-1}^2}\left(1 - \frac{1}{\lambda_{p_0}^2}\right) = \frac{\alpha - p_0}{p_0}\cdot\frac{1}{2\lambda_3^2\lambda_4^2}\prod_{j\in P(p_0)}\frac{1}{\lambda_j^2}\left(1 - \frac{1}{\lambda_{p_0}^2}\right).$$

□

Example 3.8. We construct the elements that intervene in any Goldbach Conjecture function $\mathfrak{G}$ where $\alpha = 18$. In this case, $\alpha/2 = 9,\ \alpha/2 - 1 = 8,\ \alpha/2 - 3 = 6$. Then, the coefficients $\xi_2,\ \xi_3,\ \xi_4,\ \xi_5$ can be thus expressed

$$\xi_2 > 0\ ,\ \xi_3 = \lambda_3\xi_2\ ,\ \xi_4 = \lambda_4\xi_3\ ,\ \xi_5 = \lambda_5\xi_4 \quad (\lambda_i > 1).$$

Then, $x_4, x_5, \ldots, x_{11}$ are readily determined. Using (2.2) and (2.3) we obtain the expression of x_i for i natural number $(4 \le i \le 11)$,

$$x_4 = \xi_2^2/2 = x_5 \text{ (5 prime)},$$
$$x_6 = \xi_2\xi_3 - \xi_2^2/2 = (\lambda_3 - 1/2)\xi_2^2 = x_7 \text{ (7 prime)},$$

$$x_8 = \xi_2\xi_4 - \xi_2^2/2 = (\lambda_4\lambda_3 - 1/2)\xi_2^2.$$

Considering that $|y_j| = x_{\alpha-j-1}$ $(\forall j \in \mathbb{N} :\ 4 \leq j \leq \alpha/2 - 1)$

$$x_9 = |y_8| = \xi_2\xi_4 - \xi_2\xi_3 + \xi_3^2/2 = (\lambda_4\lambda_3 - \lambda_3 + \lambda_3^2/2)\xi_2^2,$$

$$x_{10} = |y_7| = \xi_2\xi_5 - \xi_2\xi_3 + \xi_3^2/2 = (\lambda_5\lambda_4\lambda_3 - \lambda_3 + \lambda_3^2/2)\xi_2^2$$
$$= x_{11} = |y_6| \text{ (11 prime)},$$
$$\xi_6^2 = (x_6/x_5)\xi_5^2 = 2(\lambda_3 - 1/2)\lambda_5^2\lambda_4^2\lambda_3^2\xi_2^2.$$

Now x_{12} and x_{13} are readily determined

$$x_{12} = |y_5| = \xi_2\xi_6 - \xi_2\xi_4 + \xi_3\xi_4 - \xi_3^2/2$$
$$= (\lambda_5\lambda_4\lambda_3\sqrt{2(\lambda_3 - 1/2)} - \lambda_4\lambda_3 + \lambda_4\lambda_3^2 - \lambda_3^2/2)\xi_2^2 = x_{13} = |y_4| \text{ (13 prime)}.$$

We have $\xi_7^2 = \lambda_7^2\xi_6^2$ and $\xi_8^2 = (x_8/x_7)\xi_7^2$. Besides

$$F_5 = \frac{\alpha - 5}{5} \cdot \frac{1}{2\lambda_3^2\lambda_4^2}\left(1 - \frac{1}{\lambda_5^2}\right),\quad F_7 = \frac{\alpha - 7}{7} \cdot \frac{1}{2\lambda_3^2\lambda_4^2\lambda_5^2}\left(1 - \frac{1}{\lambda_7^2}\right).$$

Choosing at random $\xi_9^2 > 0$, the remaining coefficients are readily determined

$$\xi_{10}^2 = \xi_{\alpha-8}^2 = \frac{|y_7|}{|y_8|}\xi_9^2,$$
$$\xi_{11}^2 = \xi_{\alpha-7}^2 = |y_6|\ (\ |y_7|\xi_{\alpha-8}^{-2} + F_7\ )^{-1} = |y_6|\ (\ |y_8|\xi_9^{-2} + F_7\ )^{-1},$$
$$\xi_{12}^2 = \xi_{\alpha-6}^2 = \frac{|y_5|}{|y_6|}\xi_{\alpha-7}^2 = |y_5|\ (\ |y_8|\xi_9^{-2} + F_7\ )^{-1},$$
$$\xi_{13}^2 = \xi_{\alpha-5}^2 = |y_4|\ (\ |y_5|\xi_{\alpha-6}^{-2} + F_5\ )^{-1} = |y_4|\ (\ |y_8|\xi_9^{-2} + F_5 + F_7\ )^{-1}.$$

For $\xi_2 \in (0, +\infty),\ \lambda_j \in (1, +\infty),\ \xi_9 \in (0, +\infty)$, we obtain all the Goldbach Conjecture functions $\mathfrak{G}$ associated to $\alpha = 18 :\ \mathfrak{G} = \mathfrak{G}_{(18,\, \xi_2,\, \xi_9,\, \bar{\lambda})}$ with $\bar{\lambda} = (\lambda_3, \lambda_4, \lambda_5, \lambda_7)$.

Theorem 3.9. *Let α be an even number $(\alpha \geq 16)$, and $\mathfrak{G}_{(\alpha,\, \xi_2,\, \xi_{\alpha/2},\, \bar{\lambda})}$ be any Goldbach Conjecture function associated to α. Denote $n(\alpha) := \#(\mathcal{J})$ $(\mathcal{J} = \{3, 4\} \cup \mathfrak{P})$. Then, there exist functions*

$$f_i,\ g_j,\ h_j : (1, +\infty)^{n(\alpha)} \to \mathbb{R}$$

with $i \in \mathbb{N},\ j \in \mathbb{N},\ 2 \leq i \leq \alpha/2 - 3,\ 4 \leq j \leq \alpha/2 - 1$ such that

i) $\xi_i^2 = f_i(\bar{\lambda})\xi_2^2$. *ii)* $x_j = g_j(\bar{\lambda})\xi_2^2$. *iii)* $|y_j| = h_j(\bar{\lambda})\xi_2^2$.

Proof. Considering that $|y_j| = x_{\alpha-j-1}$ $(\forall j \in \mathbb{N} : 4 \leq j \leq \alpha/2-1)$ is sufficient to prove that there exist functions

$$f_i,\ g_j : (1, +\infty)^{n(\alpha)} \to \mathbb{R}\ \ (i \in \mathbb{N},\ j \in \mathbb{N},\ 2 \leq i \leq \alpha/2 - 3,\ 4 \leq j \leq \alpha - 5)$$

such that $i)'\ \xi_i^2 = f_i(\overline{\lambda})\xi_2^2,\ ii)'\ x_j = g_j(\overline{\lambda})\xi_2^2$. Then we would choose

$$h_j = g_{\alpha-j-1}\ \ (j \in \mathbb{N},\ 4 \leq j \leq \alpha/2 - 1)\,.$$

Following example, 3.8, $i)'$ and $ii)'$ are true for the natural numbers i, j where $2 \leq i \leq 5,\ 4 \leq j \leq 11$, that is, $i)'$ and $ii)'$ are true for every ξ_i^2, x_j naturally associated to the prime $p_0 = 5$. Now, regardless of the mentioned example, we prove that $i)$ and $ii)$ are true for the natural numbers i, j where $6 \leq i \leq 7,\ 12 \leq j \leq 13$. In fact,

$$\xi_6^2 = \frac{x_6}{x_5}\xi_5^2 = \frac{g_6(\bar\lambda)}{g_5(\bar\lambda)} f_5(\bar\lambda)\xi_2^2 = f_6(\bar\lambda)\xi_2^2\ , \text{ if we define } f_6 = (g_6/g_5)f_5.$$

The addends that appear in x_{12} and x_{13} have the form $\pm\ \xi_l\xi_k$ or $\pm\ \xi_h^2/2$, (l, k, h natural numbers where $2 \leq l \leq 6,\ 2 \leq k \leq 6,\ 2 \leq h \leq 6$), that is, we have addends of the form

$$\pm\sqrt{f_l(\bar\lambda)f_k(\bar\lambda)}\xi_2^2 \text{ or } \pm f_h(\bar\lambda)\xi_2^2/2.$$

Thus, x_{12} and x_{13} can be written $x_{12} = g_{12}(\bar\lambda)\xi_2^2,\ x_{13} = g_{13}(\bar\lambda)\xi_2^2$. Now, $\xi_7^2 = \lambda_7^2\xi_6^2 = \lambda_7^2 f_6(\bar\lambda)\xi_2^2 = f_7(\bar\lambda)\xi_2^2$ (if we define $f_7 = \lambda_7^2 f_6$) then, x_{14} and x_{15} are readily determined and their addends have the form $\pm\ \xi_l\xi_k$ or $\pm\ \xi_h^2/2$ (l, k, h natural numbers where $2 \leq l \leq 7,\ 2 \leq k \leq 7,\ 2 \leq h \leq 7$).

Following the reasoning stated above, x_{14} and x_{15} can be expressed $x_{14} = g_{14}(\bar\lambda)\xi_2^2,\ x_{15} = g_{15}(\bar\lambda)\xi_2^2$. We now consider the prime p_0 (where $7 < p_0 \leq s_0$). Following the previous outline we easily prove that if $i)'$ and $ii)'$ are true for every i, j natural numbers (where $2 \leq i \leq \gamma(p_0),\ 4 \leq j \leq 2\gamma(p_0) + 1$) then $i)'$ and $ii)'$ are also true for every i, j natural numbers where $2 \leq i \leq p_0,\ 4 \leq j \leq 2p_o + 1$, being irrelevant whether $s_0 < \alpha/2 - 3$ or not.

$\square$

Corollary 3.10. Let α be an even number ($\alpha \geq 16$), and be $\mathfrak{G}_{(\alpha,\, \xi_2, \xi_{\alpha/2},\, \bar\lambda)}$ any Goldbach Conjecture function associated to α. Then, $|x_{k_0-1}|\ |x_{k_0}|^{-1}$ and $|y_{k_0-1}|\ |y_{k_0}|^{-1}$ do not depend on ξ_2^2, $\forall k_0$ natural number, $5 \leq k_0 \leq \alpha/2 - 1$.

Definition 3.11. Let α be an even number ($\alpha \geq 16$), $\mathfrak{G} = \mathfrak{G}_{(\alpha,\, \xi_2, \xi_{\alpha/2},\, \bar\lambda)}$ any Goldbach Conjecture function associated to α. If $\lambda_i = u \in (1, +\infty), \forall i \in \{3, 4\} \cup \mathfrak{P}$, we say that $\mathfrak{G}$ is a *scalar Goldbach Conjecture function associated to* α. We denote such a function by $\mathfrak{G} = \mathfrak{G}_{(\alpha,\, \xi_2,\, \xi_{\alpha/2},\, u)}$.

Theorem 3.12. *Let α be an even number ($\alpha \geq 16$), $\mathfrak{G} = \mathfrak{G}_{(\alpha,\, \xi_2,\, \xi_{\alpha/2},\, u)}$ any Goldbach Conjecture function associated to α. Then for every $k_0 \in \mathbb{N}$ with $4 \leq k_0 \leq \alpha - 5$ we verify $\lim_{u \to 1^+} x_{k_0} = \xi_2^2/2$.*

Proof. We readily determine $x_4, x_5, \ldots x_{11}$ choosing

$$\xi_3^2 = \lambda_3^2\xi_2^2 = u^2\xi_2^2, \quad \xi_4^2 = \lambda_3^2\lambda_4^2\xi_2^2 = u^4\xi_2^2, \quad \xi_5^2 = \lambda_3^2\lambda_4^2\lambda_5^2\xi_2^2 = u^6\xi_2^2.$$

Following the example 3.8

$$x_4 = x_5 = \xi_2^2/2, \quad x_6 = x_7 = (u - 1/2)\xi_2^2, \quad x_8 = (u^2 - 1/2)\xi_2^2,$$
$$x_9 = (3u^2/2 - u)\xi_2^2, \quad x_{10} = x_{11} = (u^3 + u^2/2 - u)\xi_2^2.$$

Therefore we verify $\lim_{u\to 1^+} x_{k_0} = \xi_2^2/2$ for every $k_0 \in \mathbb{N}$ with $4 \le k_0 \le 11$. Now, $\xi_6^2 = (x_6/x_5)\xi_5^2$ thus

$$\lim_{u\to 1^+} \xi_6^2 = \lim_{u\to 1^+} \frac{x_6}{x_5} u^6 \xi_2^2 = \xi_2^2.$$

We have readily determined x_{12} and x_{13}. Following (2.4), for any scalar Goldbach $\mathfrak{G}_{(\alpha,\, \xi_2,\, \xi_{\alpha/2},\, u)}$ and for every natural number k_0 with $12 \le k_0 \le \alpha - 5$ and $\alpha \ge 18$ the expression of x_{k_0} is (if $\lfloor \sqrt{k_0} \rfloor = \lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor$)

$$\begin{aligned} x_{k_0} = \xi_2(\xi_{\lfloor k_0/2\rfloor} - \xi_{\lfloor k_0/3\rfloor}) + \xi_3(\xi_{\lfloor k_0/3\rfloor} - \xi_{\lfloor k_0/4\rfloor}) + \ldots + \\ \xi_{\lfloor \sqrt{k_0}\rfloor - 1}(\, \xi_{\lfloor k_0/(\lfloor \sqrt{k_0}\rfloor - 1)\rfloor} - \xi_{\lfloor k_0/\lfloor \sqrt{k_0}\rfloor \rfloor}\,) + (1/2)\xi^2_{\lfloor \sqrt{k_0}\rfloor}. \end{aligned}$$

If $\lfloor \sqrt{k_0} \rfloor < \lfloor k_0/\lfloor \sqrt{k_0} \rfloor \rfloor$ the expression of x_{k_0} is

$$\begin{aligned} x_{k_0} = \xi_2(\xi_{\lfloor k_0/2\rfloor} - \xi_{\lfloor k_0/3\rfloor}) + \xi_3(\xi_{\lfloor k_0/3\rfloor} - \xi_{\lfloor k_0/4\rfloor}) + \ldots + \\ \xi_{\lfloor \sqrt{k_0}\rfloor}(\, \xi_{\lfloor k_0/\lfloor \sqrt{k_0}\rfloor \rfloor} - (1/2)\xi_{\lfloor \sqrt{k_0}\rfloor}\,). \end{aligned}$$

Considering that $\lim_{u\to 1^+} \xi_i^2 = \xi_2^2$ $(2 \le i \le 6)$ we conclude $\lim_{u\to 1^+} x_{12} = \lim_{u\to 1^+} x_{13} = \xi_2^2/2$. Now, $\xi_7^2 = \lambda_7^2 \xi_6^2 = u^2 \xi_6^2$ thus, $\lim_{u\to 1^+} \xi_7^2 = \xi_2^2$. According to the construction of $\mathfrak{G}$ and by a simple induction process we obtain

$$\lim_{u\to 1^+} x_{k_0} = \xi_2^2/2 \quad (\forall k_0 \in \mathbb{N},\ 2 \le k_0 \le \alpha - 5).$$

$\square$

Corollary 3.13. Since $|y_j| = x_{\alpha-j-1}$ $(\forall j \in \mathbb{N},\ 4 \le j \le \alpha/2 - 1)$, then, $\lim\limits_{u\to 1^+} x_{k_0} = \lim\limits_{u\to 1^+} |y_{k_0}| = \xi_2^2/2 \quad (\forall k_0 \in \mathbb{N},\ 4 \le k_0 \le \alpha/2 - 1)$.

3.2. **Dynamic processes associated to $\mathbb{N}$.**

Theorem 3.14. *The following set is infinite:*

$$A = \{\alpha \in \mathbb{N} : \alpha \textit{ even},\ \alpha \ge 16, (\alpha/2 \textit{ and } \alpha - 3 \textit{ composite})\}.$$

Proof. Consider $A_1 = \{12k : k \in \mathbb{N}, k \ge 2\}$. Obviously $12k$ is even, $\alpha \ge 16$, $12k/2 = 6k$ is composite and $12k - 3 = 3(4k-1)$ is also composite, thus $A_1 \subset A$ and A_1 is infinite. As a consequence, A is an infinite set. $\square$

Choose $\xi_2^2 = \xi_{\alpha/2}^2 = 1$, fix $(\alpha, u) \in A \times (1, +\infty)$ and denote $\mathfrak{G}_{(\alpha,1,1,u)}$ by $\mathfrak{G}$. Consider the continuous function $\mathfrak{G} : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}$. From it we construct the functions $v, s : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}$

$$v(\hat{k}) = \int_{\hat{4}}^{\hat{k}} \mathfrak{G}(\tau)\, d\tau, \quad s(\hat{k}) = \int_{\hat{4}}^{\hat{k}} v(\tau)\, d\tau.$$

Then $s'(\hat{k}) = v(\hat{k}), v(\hat{4}) = 0, s''(\hat{k}) = v'(\hat{k}) = \mathfrak{G}(\hat{k}), s(\hat{4}) = 0$. That is, we have constructed a family of movements with continuous acceleration $\mathfrak{G}(\hat{k})$ that depend on $u > 1$ in which each state of time $t = \hat{k}$ with $t \in [\hat{4}, \hat{\alpha} \div \hat{2}]$ is associated to the real number k $(4 \le k \le \alpha/2 - 1)$ by means of the bijection $\psi^{-1}(t) = k$. Consequentially, each natural number k_0 $(4 \le k_0 \le \alpha/2 - 1)$ is associated with time state t_{k_0}. Following the corollary 3.13 we verify

$$\lim_{u\to 1^+} x_{k_0} = 1/2, \quad \lim_{u\to 1^+} y_{k_0} = -1/2, \quad (4 \leq k_0 \leq \alpha/2 - 1).$$

Also, considering the construction of $\mathfrak{G}$, we have $\lim_{u\to 1^+} \xi_i^2 = 1$ for all $2 \leq i \leq \alpha/2-1$. This means that in the limit position, $\psi = \psi^{-1} = I$ (identity function on $[4, \alpha/2]$) and the essential points have been transformed into

$$P_{k_0} = (x_{k_0}, y_{k_0}) = (1/2, -1/2), \quad (\forall k_0 \in \mathbb{N},\ 4 \leq k_0 \leq \alpha/2 - 1).$$

In other words, the characterization 2.14 about the fact of being α the sum of two prime numbers has been lost. This leads to the following conclusion

There exists at least a characterization of the Goldbach Conjecture in an infinite set of even numbers that depends on time.

Note that we have identified instant of time with real number in the mathematical continuum constructed via Cauchy sequences or Dedekind cuts. This identification could no be possible in the Brouwer's continuum ('time is the only a priori of mathematics' [2]).

> *How then do assertions arise which concern, not all natural, but all real numbers, i.e., all values of a real variable? Brouwer shows that frequently statements of this form in traditional analysis, when correctly interpreted, simply concern the totality of natural numbers. In cases where they do not, the notion of sequence changes its meaning: it no longer signifies a sequence determined by some law or other, but rather one that is created step by step by free acts of choice, and thus remains in statu nascendi. This 'becoming' selective sequence represents the continuum, or the variable, while the sequence determined ad infinitum by a law represents the individual real number falling into the continuum. The continuum no longer appears, to use Leibniz's language, as an aggregate of fixed elements but as a medium of free 'becoming'.* (H. Weyl [8])

References

[1] Tom M. Apostol, *Mathematical Analysis*, Addison-Wesley Pub. Co. (1974) pg. 421.
[2] L.E.J. Brouwer. *Collected works I. Philosophy and Foundations of Mathematics.* North-Holland, Amsterdam, 1975.
[3] A. Lentin, J. Rivaud, *Leçons d'algèbre moderne*, Vuibert (1957) pg. 33.
[4] Fernando Revilla, *Goldbach Conjecture and Peano Arithmetic*, Transcripts of the First International Congress of Applied Mathematics (Theoretical foundations of applied mathematics) (Madrid, 2007), ref. 702, pg. 451-454.
[5] Walter Rudin, *Real and Complex Analysis*, McGraw-Hill,(1987), pg. 49-55.
[6] Michael Spivak, *Calculus*, Cambridge University Press, 2006, pg. 172 and 234.
[7] J.J. Sylvester, *On certain inequalities relating to prime numbers*, Nature 38 (1888) 259-262, y reproducida en *Collected Mathematical Papers*, Volume 4, pg. 600, Chelsea, New York, 1973.
[8] Hermann Weyl, *Philosophy of Mathematics and Natural Science*, Princeton University Press (1949) pg. 52.
[9] Yuan Wang, *The Goldbach Conjecture*, Word Scientific Publishing Co. Pte. Ltd., (2002) pg. 1.

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